Geometric thermodynamical formalism and real analyticity for meromorphic functions of finite order

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Abstract. Working with well chosen Riemannian metrics and employing Nevanlinna's theory, we make the thermodynamical formalism work for a wide class of hyperbolic meromorphic functions of finite order (including in particular exponential family, elliptic functions, cosine, tangent and the cosine–root family and also compositions of these functions with arbitrary polynomials). In particular, the existence of conformal (Gibbs) measures is established and then the existence of probability invariant measures equivalent to conformal measures is proven. As a geometric consequence of the developed thermodynamic formalism, a version of Bowen's formula expressing the Hausdorff dimension of the radial Julia set as the zero of the pressure function and, moreover, the real analyticity of this dimension, is proved.

1. Introduction

One of the most fruitful tool in the study of ergodic, stochastic or geometric properties of a holomorphic dynamical system is the thermodynamical formalism. We present a completely new uniform approach that makes this theory available for a very wide class of meromorphic functions of finite order. The key point is that we associate to a given meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ a suitable Riemannian metric $d\sigma = \gamma |dz|$. We then use Nevanlinna's theory to construct conformal measures for the potentials $-t \log |f'|_\sigma$ and to control the corresponding Perron–Frobenius operator's. Here

$$|f'(z)|_\sigma = |f'(z)| \frac{\gamma \circ f(z)}{\gamma(z)}$$

is the derivative of f with respect to the metric $d\sigma$. With this tool in hand we obtain then geometric information about the Julia set $J(f)$ and about the radial (or conical) Julia set

$$\mathcal{J}_r(f) = \{ z \in J(f) \ : \ \liminf_{n\to\infty} |f^n(z)| < \infty \} \ .$$

We now give a precise description of our results.

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1.1. **Thermodynamical formalism.** Various versions of thermodynamic formalism and finer fractal geometry of transcendental entire and meromorphic functions have been explored since the middle of 90's, and have speeded up since the year 2000 (see for ex. [Ba], [CS1], [CS2][KU1], [KU2], [KU3], [MyU], [UZ1], [UZ2], [UZ3], and especially the survey article [KU4] touching on most of the results obtained by now). Some interesting and important classes of functions, including exponential λe^z and elliptic, have been fairly well understood. Essentially all of them were periodic, the methods they were dealt with broke down in the lack of periodicity, and required to project the dynamics down onto the appropriate quotient space, either torus or infinite cylinder. One has actually never completely gone back to the original phase space, the complex plane $\mathbb{C}$. A nice exception is the case of critically non-recurrent elliptic functions treated in [KU2], where the special but most important potential $-\mathrm{HD}(J(f))\log|f'|$ was explored in detail. In this paper we propose an entirely different approach. We do not need periodicity and we work on the complex plane itself. The main idea, which among others allows us to abandon periodicity, is that we associate to a given meromorphic function f a Riemannian conformal metric $d\sigma = \gamma|dz|$ with respect to which the Perron-Frobenius-Ruelle (or transfer) operator

$$\mathcal{L}_t\varphi(w) = \sum_{z\in f^{-1}(w)} |f'(z)|_\sigma^{-t}\varphi(z) \tag{1.1}$$

is well defined and has all the required properties that make the thermodynamical formalism work. Such a good metric can be found for meromorphic functions $f:\mathbb{C}\to\hat{\mathbb{C}}$ that are of finite order ρ and do satisfy the following growth condition for the derivative:

Rapid derivative growth: There are $\alpha_2 > \max\{0, -\alpha_1\}$ and $\kappa > 0$ such that

$$|f'(z)| \geq \kappa^{-1}(1+|z|^{\alpha_1})(1+|f(z)|^{\alpha_2}) \tag{1.2}$$

for all $z \in J(f)\setminus f^{-1}(\infty)$. Throughout the entire paper we use the notation

$$\alpha = \alpha_1 + \alpha_2.$$

This condition is very general and forms our second main idea. It is comfortable to work with and relatively easy to verify (see Section 3) for a large natural class of functions which include the entire exponential family λe^z, certain other periodic functions ($sin(az+b)$, $\lambda\tan(z)$, elliptic functions...), the cosine-root family $\cos(\sqrt{az+b})$ and the composition of these functions with arbitrary polynomials. Let us repeat that in Section 3 these and more examples are described in greater detail. The Riemannian metric σ we are after is

$$d\sigma(z) = (1+|z|^{\alpha_2})^{-1}|dz|.$$

Let (X, m) be a probability measure and $T: X\to X$ a measurable map. Recall that, given a bounded above non-negative measurable function $g: X\to[0,+\infty)$, the

measure m is called g-conformal provided that

$$m(T(A)) = \int_A g dm$$

for every measurable subset A of X such that $T|_X$ is injective. Our third and fourth basic ideas were to revive the old method of construction of conformal measures from [DU1] (which itself stemmed from the work of Sullivan [Su] and Patterson [Pa]) and to employ results and methods coming from Nevanlinna's theory. These allowed us to perform the construction of conformal measures and to get good control of the Perron-Frobenius-Ruelle operator, resulting in the following key result of our paper.

Theorem 1.1. *If $f : \mathbb{C} \to \hat{\mathbb{C}}$ is an arbitrary hyperbolic meromorphic function of finite order ρ that satisfies the rapid derivative growth condition (1.2), then for every $t > \frac{\rho}{\alpha}$ the following are true.*

(1) *The topological pressure* $\mathrm{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_t^n(1\!\!1)(w)$ *exists and is independent of $w \in J(f)$.*
(2) *There exists a unique $\lambda|f'|_\sigma^t$-conformal measure m_t and necessarily $\lambda = e^{\mathrm{P}(t)}$. Also, there exists a unique probability Gibbs state μ_t, i.e. μ_t is f-invariant and equivalent to m_t. Moreover, both measures are ergodic and supported on the radial (or conical) Julia set.*
(3) *The density $\psi = d\mu_t/dm_t$ is a continuous and bounded function on the Julia set $J(f)$.*

Remark 1.2. *For the existence of $e^{\mathrm{P}(t)}|f'|_\sigma^t$-conformal measures the assumption of hyperbolicity is not needed (see Section 5).*

Note that even in the context of exponential functions (λe^z) and elliptic functions, this result is new since it concerns the map f itself and not its projection onto infinite cylinder or torus.

An important case in Theorem 1.1 is when h is a zero of the pressure function $t \mapsto \mathrm{P}(t)$. In this situation, the corresponding measure m_h is $|f'|_\sigma^h$-conformal (also called simply h-conformal). We will see that such a (unique) zero $h > \rho/\alpha$ exists provided the function f satisfies a mild growth condition on the characteristic function (see (7.1)) and, most importantly the following balanced growth condition.

Balanced growth condition: There are $\alpha_2 > \max\{0, -\alpha_1\}$ and $\kappa > 0$ such that

$$\kappa^{-1}(1 + |z|^{\alpha_1})(1 + |f(z)|^{\alpha_2}) \leq |f'(z)| \leq \kappa(1 + |z|^{\alpha_1})(1 + |f(z)|^{\alpha_2}) \tag{1.3}$$

for all finite $z \in J(f) \setminus f^{-1}(\infty)$.

Hyperbolic meromorphic functions of positive and finite order that satisfy these conditions are called *dynamically regular.*

1.2. **Bowen's formula.** Starting from Section 7 we provide geometric applications of the key result above and provide, in particular, the following version of Bowen's formula.

Theorem 1.3. *(Bowen's formula) If $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a dynamically regular function, then the pressure function* $\mathrm{P}(t)$ *has a unique zero $h > \rho/\alpha$ and*

$$\mathrm{HD}(J_r(f)) = h \, .$$

This type of formulas has a long and rich history. It has appeared the first time in the classical Bowen's paper [Bw] and since then has been generalized and adopted to a vast number of contexts, taking perhaps on the most perfect form in the class of hyperbolic rational functions. In this class and in many others the zero of the pressure function is the value of the Hausdorff dimension of the entire Julia set (which is false for entire functions [UZ1]). By a reasoning, which is by now standard, Theorem 1.3 leads to the following.

Corollary 1.4. *With the assumptions of Theorem 1.3, we have* $\mathrm{HD}(J_r(f)) < 2$.

This property applied to the sine or exponential family and combined with results of McMullen [McM] (who showed that the Hausdorff dimension of these functions is always two) gives the following.

Corollary 1.5. *If f is any hyperbolic member of the exponential ($z \mapsto \lambda e^z$) or the sine ($z \mapsto sin(\alpha z + \beta)$, $\alpha \neq 0$) family then the hyperbolic dimension* $\mathrm{HD}(J_r(f))$ *is strictly less then* $\mathrm{HD}(J(f))$.

Note that such a phenomenon does not exist in the setting of rational functions. For the exponential family it has been proven in [UZ1].

Proof of Corollary 1.4. Indeed, by Theorem 1.3 and by Theorem 1.1 there exists an $|f'|_\sigma^h$-conformal measure for f. Suppose to the contrary that $h = 2$. Now the proof is standard (see [UZ1] or [My1] for details): Firstly, using the definition of the set $J_r(f)$, which gives possibility of taking pull-backs of points lying in a compact region, and applying Koebe's Distortion Theorem, one shows that the measure m_h and the 2-dimensional Lebesgue measure restricted to $J_r(f)$ are equivalent. Secondly, consider an arbitrary point $z \in J_r(f)$. As above it has infinitely many pull-backs from a compact region. Since the Julia set is "uniformly" nowhere dense on any compact part, using Koebe's Distortion Theorem, one easily deduces that z cannot be a Lebesgue density point of $J_r(f)$. Thus the Lebesgue measure of $J_r(f) = 0$, and this contradiction finishes the proof. $\square$

1.3. **Real analyticity.** Answering the conjecture of D. Sullivan, D. Ruelle in [R] (1982) gave a proof of the real-analytic dependence of the Hausdorff dimension of the Julia set for hyperbolic rational maps. More recently, this fact was extended in [UZ2, CS2] to some special families of meromorphic functions (in particular the exponential family). It was shown that the variation of the Hausdorff dimension of the radial Julia set $\mathcal{J}_r(f)$ is real-analytic at hyperbolic functions. Note that in the case of hyperbolic rational functions the Julia and the radial Julia set coincide. This is no longer true in the meromorphic setting and, as we have seen in Corollary 1.5, there is often a gap between the *hyperbolic dimension,* i.e. the Hausdorff dimension of the radial Julia set, and the Hausdorff dimension of the Julia set itself [UZ1].

We investigate the variation of the hyperbolic dimension of meromorphic functions in a very general setting and prove in particular the following result which contains as special cases the real analyticity facts established in [UZ2] and [CS2].

Theorem 1.6. *Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be either the sine, tangent, exponential or the Weierstrass elliptic function and let $f_\lambda(z) = f(\lambda_d z^d + \lambda_{d-1} z^{d-1} + \dots + \lambda_0)$, $\lambda = (\lambda_d, \lambda_{d-1}, \dots, \lambda_0) \in \mathbb{C}^* \times \mathbb{C}^d$. Then the function*

$$\lambda \mapsto \mathrm{HD}(\mathcal{J}_r(f_\lambda))$$

is real-analytic in a neighbourhood of each parameter λ^0 giving rise to a hyperbolic function f_{λ^0}.

This result is an example of an application of the general Theorem 1.7 (via Theorem 10.1) that we present now.

The Speiser class $\mathcal{S}$ is the set of meromorphic functions $f : \mathbb{C} \to \hat{\mathbb{C}}$ that have a finite set of singular values $sing(f^{-1})$. We will work in the subclass $\mathcal{S}_0$ which consists in the functions $f \in \mathcal{S}$ that have a strictly positive and finite order $\rho = \rho(f)$ and that are of divergence type. Fix Λ, an open subset of $\mathbb{C}^N$, $N \geq 1$. Let

$$\mathcal{M}_\Lambda = \{f_\lambda \in \mathcal{S}_0 \,;\, \lambda \in \Lambda\} \ , \ \ \Lambda \subset \mathbb{C}^N,$$

be a holomorphic family such that the singular points $sing(f_\lambda^{-1}) = \{a_{1,\lambda}, \dots, a_{d,\lambda})$ depend continuously on $\lambda \in \Lambda$. Consider furthermore $\mathcal{H} \subset \mathcal{S}_0$, the set of hyperbolic functions from $\mathcal{S}_0$ and put

$$\mathcal{H}\mathcal{M}_\Lambda = \mathcal{M}_\Lambda \cap \mathcal{H}.$$

We say that $\mathcal{M}_\Lambda$ *is of bounded deformation* if there is $M > 0$ such that for all $j = 1, \dots, N$

$$\left| \frac{\partial f_\lambda(z)}{\partial \lambda_j} \right| \leq M |f_\lambda'(z)| \ \ , \quad \lambda \in \Lambda \ \ and \ \ z \in \mathcal{J}(f_\lambda). \tag{1.4}$$

We also say that $\mathcal{M}_\Lambda$ is *uniformly balanced* provided every $f \in \mathcal{M}_\Lambda$ satisfies the condition (1.3) with some fixed constants $\kappa, \alpha_1, \alpha_2$.

Theorem 1.7. *Suppose $f_{\lambda^0} \in \mathcal{HM}_\Lambda$ is dynamically regular and that $U \subset \Lambda$ is an open neighborhood of λ^0 such that $\mathcal{M}_U$ is uniformly balanced with $\alpha_1 \geq 0$ and of bounded deformation. Then the map*

$$\lambda \mapsto \mathrm{HD}(\mathcal{J}_r(f_\lambda))$$

is real-analytic near λ^0.

2. Generalities

The reader may consult, for example, [Nev1], [Nev2] or [H] for a detailed exposition on meromorphic functions and [Bw] for their dynamical aspects. We collect here the properties of interest for our concerns. The Julia set of a meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ is denoted by $\hat{J}(f)$ and the Fatou set by $\mathcal{F}_f$. Since we always work in the finite plane we denote $J(f) = \hat{J}(f) \cap \mathbb{C}$. By Picard's theorem, there are at most two points $z_0 \in \hat{\mathbb{C}}$ that have finite backward orbit $\mathcal{O}^-(z_0) = \bigcup_{n\geq 0} f^{-n}(z_0)$. The set of these points is the exceptional set $\mathcal{E}_f$. In contrast to the situation of rational maps it may happen that $\mathcal{E}_f \subset J(f)$. Iversen's theorem [Iv, Nev1] asserts that every $z_0 \in \mathcal{E}_f$ is an asymptotic value. Consequently, $\mathcal{E}_f \subset sing(f^{-1})$ the set of critical and finite asymptotic values. The post-critical set $\mathcal{P}_f$ is defined to be the closure in the plane of

$$\bigcup_{n\geq 0} f^n\big(sing(f^{-1}) \setminus f^{-n}(\infty)\big) \ .$$

Let us introduce the following definitions.

Definition 2.1. *A meromorphic function f is called topologically hyperbolic if*

$$\delta(f) := \frac{1}{4}\mathrm{dist}\big(J(f), \mathcal{P}_f\big) > 0.$$

and it is called expanding if there is $c > 0$ and $\lambda > 1$ such that

$$|(f^n)'(z)| \geq c\lambda^n \quad for\ all\ \ z \in J(f) \setminus f^{-n}(\infty) \ .$$

A topologically hyperbolic and expanding function is called hyperbolic.

The Julia set of a hyperbolic function is never the whole sphere. We thus may and we do assume that the origin $0 \in \mathcal{F}_f$ is in the Fatou set (otherwise it suffices to conjugate the map by a translation). This means that there exists $T > 0$ such that

$$D(0, T) \cap J(f) = \emptyset. \tag{2.1}$$

The derivative growth condition (1.2) can then be reformulated in the following more convenient form:

There are $\alpha_2 > 0$, $\alpha_1 > -\alpha_2$ and $\kappa > 0$ such that

$$|f'(z)| \geq \kappa^{-1}|z|^{\alpha_1}|f(z)|^{\alpha_2} \quad for\ all\ \ z \in J(f) \setminus f^{-1}(\infty) \ . \tag{2.2}$$

Similarly, the balanced condition (1.3) becomes

$$\kappa^{-1}|z|^{\alpha_1}|f(z)|^{\alpha_2} \leq |f'(z)| \leq \kappa|z|^{\alpha_1}|f(z)|^{\alpha_2} \quad for\ all\ \ z \in J(f) \setminus f^{-1}(\infty) \tag{2.3}$$

and the metric $d\sigma(z) = |z|^{-\alpha_2}|dz|$.

It is well known that in the context of rational functions topological hyperbolicity and expanding property are equivalent. Neither implication is established for transcendental functions. However, under the rapid derivative growth condition (2.2) with $\alpha_1 \geq 0$ topological hyperbolicity implies hyperbolicity.

Proposition 2.2. *Every topologically hyperbolic meromorphic function satisfying the rapid derivative growth condition with* $\alpha_1 \geq 0$ *is expanding, and consequently, hyperbolic.*

Proof. Let us fix $\lambda \geq 2$ such that $\lambda\kappa^{-1}T^\alpha \geq 2$. In view of rapid derivative growth (2.2) and (2.1)

$$|f'(z)| \geq \kappa^{-1}T^\alpha \quad \textit{for all} \;\; z \in J(f) \tag{2.4}$$

and

$$|f'(z)| \geq \lambda \quad \textit{for all} \;\; z \in f^{-1}(J(f) \setminus D(0,R)) \tag{2.5}$$

provided $R > 0$ has been chosen sufficiently large. In addition we need the following.

Claim: There exists $p = p(\lambda, R) \geq 1$ such that

$$|(f^n)'(z)| \geq \lambda \quad \textit{for all} \;\; n \geq p \;\; \textit{and} \;\; z \in \overline{D}(0,R) \cap J(f).$$

Indeed, suppose on the contrary that there is $R > 0$ such that for some $n_p \to \infty$ and $z_p \in \overline{D}(0,R) \cap J(f)$ we have

$$|(f^{n_p})'(z_p)| < \lambda. \tag{2.6}$$

Put $\delta = \delta(f)$. Then for every $p \geq 1$ there exists a unique holomorphic branch $f_*^{-n_p} : D\big(f^{n_p}(z_p), 2\delta\big) \to \mathbb{C}$ of f^{-n_p} sending $f^{n_p}(z_p)$ to z_p. It follows from $\frac{1}{4}$-Koebe's Distortion Theorem and (2.6) that

$$f_*^{-n_p}\Big(D\big(f^{n_p}(z_p), 2\delta\big)\Big) \supset D\big(z_p, \delta/(2\lambda)\big) \tag{2.7}$$

or, equivalently, that $f^{n_p}(D\big(z_p, \delta/(2\lambda)\big)) \subset D\big(f^{n_p}(z_p), 2\delta\big)$. Passing to a subsequence we may assume without loss of generality that the sequence $\{z_p\}_{p=1}^\infty$ converges to a point $z \in \overline{D}(0,R) \cap J(f)$. Since $D(\mathcal{P}_f, 2\delta) \cap D\big(f^{n_p}(z_p), 2\delta\big) = \emptyset$ for every $p \geq 1$, it follows from Montel's theorem that the family $\big\{f^{n_p}|_{D(z,(2\lambda)^{-1}\delta)}\big\}_{p=1}^\infty$ is normal, contrary to the fact that $z \in J(f)$. The claim is proved.

Let $p = p(\lambda, R) \geq 1$ be the number produced by the claim. It remains to show that

$$|(f^{2p})'(z)| \geq 2 > 1 \quad \textit{for every} \;\; z \in J(f).$$

This formula holds if $|f^j(z)| > R$ for $j = 0, 1, ..., p$ because of (2.4), (2.5) and the choice of λ. If $|z| > R$ but $|f^j(z)| \leq R$ for some $0 \leq j \leq p$, the conclusion follows from (2.4) and the claim. $\square$

The class of Speiser $\mathcal{S}$ consists in the functions f that have a finite set of singular values $sing(f^{-1})$. The classification of the periodic Fatou components is the same as the one of rational functions because any map of $\mathcal{S}$ has no wandering nor Baker domains [Bw]. Consequently, if $f \in \mathcal{S}$ then f is topologically hyperbolic if and only if the orbit of every singular value converges to one of the finitely many attracting cycles of f. This last property is stable under perturbation, a fact that we use in Section 9 and also in the next remark:

Fact 2.3. *Let $f_{\lambda^0} \in \mathcal{H}$ be a hyperbolic function and $U \subset \Lambda$ an open neighborhood of λ^0 such that, for every $\lambda \in U$, f_λ satisfies the balanced growth condition (2.3) with $\kappa > 0, \alpha_1 \geq 0$ and $\alpha_2 > 0$ independent of $\lambda \in U$. Then, replacing U by some smaller neighborhood if necessary, all the f_λ satisfy the expanding property for some c, ρ independent of $\lambda \in U$.*

We end this part by giving a more detailed description of the divergence type functions than the one given in the introduction. For a meromorphic function f of finite order ρ a theorem of Borel states that the series

$$\Sigma(t,w) = \sum_{z \in f^{-1}(w)} |z|^{-t} \tag{2.8}$$

has the exponent of convergence equal to ρ meaning that it diverges if $t < \rho$ and converges if $t > \rho$. Concerning the behavior of $\Sigma(t,w)$ in the critical case $t = \rho$ it turns out that, if $\Sigma(\rho, w) = \infty$ for some $w \in \hat{\mathbb{C}}$, then this series diverges for all but at most two values $w \in \hat{\mathbb{C}}$ (see Remark 4.6).

Definition 2.4. *If $\Sigma(\rho, w) = \infty$ for some $w \in \hat{\mathbb{C}} \setminus \mathcal{E}_f$, then the function f is said to be of divergence type.*

The symbols $\asymp$ and $\preceq$ will signify through the whole text that equality respectively inequality holds up to a multiplicative constant that is independent of the involved variables.

3. Functions that satisfy the growth condition

Here we first explain the meaning of the exponents α_1, α_2 and then we present various examples that fit into our context.

3.1. **The signification of the exponents α_1, α_2.** For entire functions the balanced growth condition (2.3) is in fact a condition on the logarithmic derivative of the function. Indeed, for all known balanced entire functions one has $\alpha_2 = 1$ and $\alpha_1 = \rho - 1$ with, as usual, ρ being the order of the function. The balanced growth condition signifies then that the logarithmic derivative of the function is of polynomial growth of order $\rho - 1$. The lemma to follow indicates that this is a general fact. Let $\mathcal{B}$ be the class of functions that have a bounded singular set $sing(f^{-1})$. Clearly $\mathcal{S} \subset \mathcal{B}$.

Lemma 3.1. *Suppose that f is a balanced entire function of class $\mathcal{B}$ and of positive finite order ρ. Then $\alpha_2 = 1$ and $\alpha_1 = \rho - 1$.*

Notice also that for all these functions the critical exponent

$$\Theta = \frac{\rho}{\alpha} = \frac{\rho}{\alpha_1 + \alpha_2} = 1.$$

We recall that the transfer operator is defined only for $t > \Theta$ and that this number is the lower bound given in Bowen's formula. Consequently, entire functions of class $\mathcal{B}$ that satisfy the conditions of Bowen's formula do have a hyperbolic dimension strictly larger then 1.

Proof of Lemma 3.1. Let f be an entire function as discribed in Lemma 3.1. Based on Wiman-Valiron theory, Eremenko [Er1] constructed an orbit $z_{n+1} = f(z_n) \in I(f)$ with

$$I(f) = \{z \in \mathbb{C}\ ,\ \lim_{n\to\infty} f^n(z) = \infty\}$$

the escaping set of f. Notice that for entire functions of class $\mathcal{B}$ we have $I(f) \subset J(f)$ ([EL]) and thus $z_n \in J(f)$ for every $n \geq 0$. This orbit has further properties. The important ones for our purpuse are that, for every $n \geq 0$,

$$|z_{n+1}| \geq \frac{1}{2} m_f(|z_n|)$$

and

$$\frac{|f'(z_n)|}{|f(z_n)|} = \frac{\nu_f(|z_n|)}{|z_n|}|1 + \varepsilon_n| \quad \textit{where} \ \ |\varepsilon_n| < \frac{1}{2}.$$

Here $m_f(r) = \max_{|z|=r} |f(z)|$ measures the maximal growth of f and the function ν_f is the so called central index. It follows now easily from the growth properties of these functions m_f and ν_f together with

$$|f'(z_n)| \asymp |z_n|^{\alpha_1} |z_{n+1}|^{\alpha_2}$$

that $\alpha_2 = 1$ and $\alpha_1 = \rho - 1$. □

For a meromorphic (non entire) function f, the exponent α_2 is determined by the multiplicity of the function at a pole. If q is the multiplicity of f at a pole b and if f satisfies the balanced growth condition (2.3) then

$$\alpha_2 = 1 + \frac{1}{q}$$

(cf. the discussion on elliptic functions below). This looks restrictive in the sense that it forces all poles to have the same multiplicity. However this problem can be overcome if one replaces the constant α_2 by a function. In order to avoid a technical more involved presentation we keep α_2 constant here.

3.2. **Examples.** First of all, the whole exponential family $f_\lambda = \lambda \exp(z)$, $\lambda \neq 0$, clearly satisfies the growth condition (2.2) with $\alpha_1 = 0$ and $\alpha_2 = 1$. More generally, if P and Q are arbitrary polynomials, then

$$f(z) = P(z)\exp(Q(z)) \quad , \; z \in \mathbb{C}\,,$$

satisfies (2.2) provided that $|f'|_{|J(f)} \geq c > 0$. In this case $\alpha_1 = deg(Q) - 1$, $\alpha_2 = 1$, the order $\rho = deg(Q)$ and consequently $\frac{\rho}{\alpha} = 1$. Assuming still that $|f'|_{|J(f)} \geq c > 0$ (which holds in particular for expanding maps), the following functions also satisfy rapid derivative growth condition (2.2):

(1) The sine family: $f(z) = \sin(az + b)$, $a, b \in \mathbb{C}$, $a \neq 0$.
(2) The cosine-root family: $f(z) = \cos(\sqrt{az + b})$ with again $a, b \in \mathbb{C}$, $a \neq 0$. Note that here $\alpha_1 = -\frac{1}{2}$ and $\alpha_2 = 1$ which explains that negative values of α_1 should be considered in (2.2).
(3) Certain solutions of Ricatti differential equations like, for example, the tangent family $f(z) = \lambda \tan(z)$, $\lambda \neq 0$, and, more generally, the functions

$$f(z) = \frac{Ae^{2z^k} + B}{Ce^{2z^k} + D} \quad \textit{with} \quad AD - BC \neq 0\,.$$

The associated differential equations are of the form $w' = kz^{k-1}(a + bw + cw^2)$ which explains that here $\alpha_1 = k - 1$ and $\alpha_2 = 2$.
(4) All elliptic functions.
(5) Any composition of one of the above functions with a polynomial.

The assertion on elliptic functions deserves some explanation. Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a doubly periodic meromorphic function and let $U = \{z \in \mathbb{C} : |z| > R\} \cup \{\infty\}$, where $R > 0$ is chosen sufficiently large so that:

a) every component V_b of $f^{-1}(U)$ is a bounded topological disc, and
b) there is $\kappa > 0$ such that for every pole b and any $z \in V_b \setminus \{b\}$ we have $|f'(z)| \asymp |f(z)|^{1+\frac{1}{q_b}}$ where q_b is the multiplicity of the pole b.

From the periodicity of f and the assumption $|f'|_{|J(f)} \geq c > 0$ easily follows now that f satisfies (2.2) with $\alpha_1 = 0$ and

$$\alpha_2 = \inf\left\{1 + \frac{1}{q_b} : b \in f^{-1}(\infty)\right\} \quad .$$

More generally, the preceding discussion shows that for any function f that has at least one pole one always has

$$\alpha_2 \leq \inf\left\{1 + \frac{1}{q_b} : b \in f^{-1}(\infty)\right\}.$$

The stronger balanced growth condition (2.3) is also satisfied by an elliptic function provided all its poles have the same order. General elliptic and meromorphic functions with poles of different order can not be of balanced growth. As already mentioned, this problem can be overcome if one allows a_2 being variable; we clarify this situation in a forthcoming paper (were also other classes of balanced functions will be given).

Uniform balanced growth is verified by various families. Here are some examples.

Lemma 3.2. *Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be either the sine, tangent, exponential or the Weierstrass elliptic function and let $f_\lambda(z) = f(\lambda_d z^d + \lambda_{d-1} z^{d-1} + ... + \lambda_0)$, $\lambda = (\lambda_d, \lambda_{d-1}, ..., \lambda_0) \in \mathbb{C}^* \times \mathbb{C}^d$. Suppose λ^0 is a parameter such that f_{λ^0} is topologically hyperbolic. Then there is a neighbourhood U of λ^0 such that $\mathcal{M}_U = \{f_\lambda \,; \lambda \in U\}$ is uniformly balanced.*

Proof. All the functions f mentioned have only finitely many singular values, they are in the Speiser class. The function f_{λ^0} being in addition topologically hyperbolic, its singular values are attracted by attracting cycles. As we already remarked in the previous section, this is a stable property in the sense that there is a neighbourhood U of λ^0 such that all the functions of $\mathcal{M}_U = \{f_\lambda \,; \lambda \in U\}$ have the same property. In particular, no critical point of f_λ is in $J(f_\lambda)$. The function f satisfies a differential equation of the form

$$(f')^p = Q \circ f$$

with Q a polynomial whose zeros are contained in $sing(f^{-1})$. For example, in the case when f is the Weierstrass elliptic function then

$$(f')^2 = 4(f - e_1)(f - e_2)(f - e_3)$$

with e_1, e_2, e_3 the critical values of f. Let $\lambda \in U$ and denote $P_\lambda(z) = \lambda_d z^d + \lambda_{d-1} z^{d-1} + ... + \lambda_0$. Since

$$(f_\lambda')^p = (f' \circ P_\lambda \, P_\lambda')^p = Q \circ f_\lambda (P_\lambda')^p$$

and $f_\lambda(z) \neq 0$ for all $z \in J(f_\lambda)$, the polynomials P_λ' and Q do not have any zero in $J(f_\lambda)$. Consequently

$$|P_\lambda'(z)| \asymp |z|^{d-1} \quad \textit{and} \quad |Q(z)| \asymp |z|^q \quad \textit{on} \quad J(f_\lambda)$$

with $q = deg(Q)$. Moreover, restricting U if necessary, the involved constants can be chosen to be independent of $\lambda \in U$. Therefore,

$$|f_\lambda'(z)| \asymp |f_\lambda(z)|^{\frac{q}{p}} |z|^{d-1}$$

for $z \in J(f_\lambda)$ and $\lambda \in U$. We verified the uniform balanced growth condition with $\alpha_1 = d - 1$ and $\alpha_2 = \frac{q}{p}$ depending on the choice of f. In the case of the Weierstrass elliptic function one has $\alpha_2 = 3/2$. □

4. Growth condition and cohomological transfer operator

For exponential or elliptic functions one can use the periodicity to project the map onto the quotient space (torus or cylinder). This idea recently lead to many new results (see [KU4] and the reference therein). Here we replace the quotient spaces by metric spaces $(\mathbb{C}, d\sigma)$ which are much more flexible. The first and essential problem however is to find the right natural metric for a given meromorphic function. We will describe now how this can be done for meromorphic functions of finite order that satisfy the rapid derivative growth condition. Recall that we work with the metric

$$d\sigma(z) = |z|^{-\alpha_2} |dz|$$

and we set $\alpha = \alpha_1 + \alpha_2$. The derivative of a function $f : \mathbb{C} \to \hat{\mathbb{C}}$ with respect to this metric is given at a point $z \in \mathbb{C}$ by the formula

$$|f'(z)|_\sigma = \frac{d\sigma(f(z))}{d\sigma(z)} = |f'(z)| \frac{|f(z)|^{-\alpha_2}}{|z|^{-\alpha_2}} = |f'(z)||z|^{\alpha_2}|f(z)|^{-\alpha_2}.$$

We will now see that this is the right choice of the metric in order for the associated transfer operator $\mathcal{L}_t$ (with the potential $-t \log |f'|_\sigma$) to act continuously on the Banach space $C_b(J(f))$ of bounded continuous functions on $J(f)$. Indeed

$$\begin{aligned} \mathcal{L}_t \varphi(w) &= \sum_{z \in f^{-1}(w)} |f'(z)|_\sigma^{-t} \varphi(z) = \sum_{z \in f^{-1}(w)} |f'(z)|^{-t} |z|^{-\alpha_2 t} |f(z)|^{\alpha_2 t} \varphi(z) \\ &= |w|^{\alpha_2 t} \sum_{z \in f^{-1}(w)} |f'(z)|^{-t} |z|^{-\alpha_2 t} \varphi(z). \end{aligned}$$

So, if f satisfies (2.2), then

$$\mathcal{L}_t 1\!\!1(w) = \sum_{z \in f^{-1}(w)} |f'(z)|_\sigma^{-t} \le \kappa^t \sum_{z \in f^{-1}(w)} |z|^{-\alpha t}. \tag{4.1}$$

Now, assume that f is of finite order ρ. Then, as we noted in the introduction, a theorem of Borel states that the last series has the exponent of convergence equal to ρ for all but at most two points (the points from $\mathcal{E}_f$). Assume that $\mathcal{E}_f$ is disjoint from the Julia set $J(f)$; this is for example true if f is topologically hyperbolic. What we need is the uniform convergence of the last series in (4.1) in order to secure continuity of the operator $\mathcal{L}_t$ on the Banach space $C_b(J(f))$ of bounded continuous functions endowed with the standard supremum norm. More precisely, we need to know that, for a given $t > \rho/\alpha$, there is $M_t > 0$ such that

$$\mathcal{L}_t 1\!\!1(w) \le M_t \quad \textit{for all} \;\; w \in J(f) \; . \tag{4.2}$$

It turns out that under our assumptions this is always true:

Theorem 4.1. *Assume that $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a finite order hyperbolic meromorphic function of rapid derivative growth. Then for every $t > \rho/\alpha$, the transfer operator $\mathcal{L}_t$ is well defined and acts continuously on the Banach space $C_b(J(f))$.*

The rest of this section is devoted to the proof of this "Uniform Borel Theorem". Our proof relies on Nevanlinna's theory of value distribution. The reader can find in the modern monograph [CY] a complete account on this topic. Let f be meromorphic of finite order ρ and let $u > \rho$. We are interested in the dependence of the following series on a:

$$\Sigma(u, a) = \sum_{f(z) = a} \frac{1}{|z|^u} \; .$$

Borel's theorem states that this series converges for every non-exceptional value $a \in \hat{\mathbb{C}}$. But is this convergence uniform? To see this we investigate the error terms in the proof of Borel's theorem as given in [Nev1, p. 265] or [Nev2, p. 261]. In order to do

this we use again the fact that $0 \in \mathcal{F}_f$. In the following we use the standard notations of Nevanlinna theory. For example, $n(t,a)$ is the number of a-points of modulus at most t, $N(r,a)$ is defined by $dN(r,a) = n(r,a)/r$ and $T(r)$ is the characteristic of f (more precisely the Ahlfors-Shimizu version of it; these two different definitions of the characteristic function only differ by a bounded amount). The first main theorem (FMT) of Nevanlinna yields the following for our situation:

Corollary 4.2 (of FMT). *There is $\Xi > 0$ such that $N(r,a) \leq T(r) + \Xi$ for all $a \in J(f)$.*

Proof. FMT as stated in [Er2] or in [H, p. 216] yields

$$N(r,a) \leq T(r) + m(0,a) \quad for\ all\ r > 0\ and\ a \in \hat{\mathbb{C}}$$

with $m(0,a) = -\log[f(0),a]$ and where $[a,b]$ denotes the chordal distance on the Riemann sphere (with in particular $[a,b] \leq 1$ for all $a,b \in \hat{\mathbb{C}}$). Since $f(0) \in \mathcal{F}_f$, there is $\tau > 0$ such that $[a,f(0)] > \tau$ for all $a \in J(f)$. It follows that the error term is bounded by

$$0 \leq m(0,a) \leq -\log\tau = \Xi \quad for\ all\ \ a \in J(f)\ .$$

□

From the second main theorem (SMT) of Nevanlinna we need the following version which is from [Nev1, p. 257] ([Nev2, p. 255] or again [H]) and which is valid only since f is supposed to be of finite order.

Corollary 4.3 (of SMT). *Let $a_1, a_2, a_3 \in \hat{\mathbb{C}}$ be distinct points. Then*

$$T(r) \leq \sum_{j=1}^{3} N(r,a_j) + S(r) \quad for\ every\ \ r > 0\ \ with\ \ S(r) = \mathcal{O}(\log(r)).$$

The error term $S(r)$ has been studied in detail and sharp estimates are known. The following results from Hinkkanen's paper [Hk] and also from Cherry-Ye's book [CY].

Lemma 4.4. *Let f be a hyperbolic meromorphic function of finite order ρ that is normalized such that $0 \in D(0,T) \subset \mathcal{F}_f$, $f(0) \notin \{0,\infty\}$ and $f'(0) \neq 0$. Then, for every $\Delta < T/4$, there exists $C_1 = C_1(\Delta) > 0$ and $C_2 > 0$ such that*

$$4N(R+\Delta,a) \geq T(R) - (3\rho+1)\log R - C_1 - C_2 \log|a|$$

for every $a \in J(f)$ and every $R > T$.

Proof. Since f is expanding there is $c > 0$ such that $|f'(z)| \geq c > 0$ for all $z \in J(f)$. Let $0 < \Delta' < \min\{\delta(f), T\}$ such that $\Delta = 2K\Delta'/c < T/4$ where K is an appropriate

Koebe distortion constant. Consider then $a \in J(f)$ and $a' \in D(a, \Delta')$. Since all the inverse branches of f are well defined on $D(a, 2\Delta')$ we have

$$n(r+\Delta, a) \geq n(r, a') \quad , \quad r > 0.$$

Consequently

$$\begin{aligned} N(R, a') &= \int_0^R \frac{n(r,a')}{r}\,dr \leq \int_0^R \frac{n(r+\Delta,a)}{r}\,dr \\ &= \int_\Delta^{R+\Delta} \frac{n(t,a)}{t}\frac{t}{t-\Delta}\,dt \leq \frac{T}{T-\Delta}\int_T^{R+\Delta}\frac{n(t,a)}{t}\,dt \\ &\leq \frac{4}{3}N(R+\Delta, a) \ \ for\ every\ \ R > T. \end{aligned}$$

Choose now $a_1, a_2, a_3 \in D(a, \Delta')$, any points that satisfy $|a_i - a_j| \geq \Delta'/3$ for all $i \neq j$. It follows then from the sharp form of SMT given in [Hk], the fact that f is of finite order, along with the normalisations stated in the lemma that

$$\begin{aligned} 4N(R+\Delta, a) &\geq \sum_{i=1}^3 N(R, a_i) \geq T(R) - S(R, a_1, a_2, a_3) \\ &\geq T(R) - (3\rho+1)\log R - C_1(\Delta) - C_2 \log|a| \end{aligned}$$

for every $a \in J(f)$ and for all $R > T$. □

We can now show the following uniform version of Borel's theorem which implies Theorem 4.1.

Proposition 4.5. *Let f be meromorphic of finite order ρ and suppose that $0 \in \mathcal{F}_f$. Then, for every $u > \rho$, there is $M_u > 0$ such that*

$$\Sigma(u, a) = \sum_{f(z)=a} \frac{1}{|z|^u} \leq M_u \quad for\ all\ \ a \in J(f)\ .$$

Proof. Recall that $J(f) \cap \overline{D}(0, T) = \emptyset$. Then $n(T, a) = N(T, a) = 0$, for all $a \in J(f)$, and by the definition of the Riemann-Stieltjes integral, integration by parts and the fact that $\lim_{r\to\infty} \frac{n(r,a)}{r^u} = 0$, we get that

$$\Sigma(u, a) = \int_T^\infty \frac{d\,n(t,a)}{t^u} = u\int_T^\infty \frac{n(t,a)}{t^{u+1}}\,dt\ .$$

In the same way

$$\int_T^\infty \frac{n(t,a)}{t^{u+1}}\,dt = u\int_T^\infty \frac{N(t,a)}{t^{u+1}}\,dt\ .$$

Putting both equations together, we get

$$\Sigma(u, a) = u^2 \int_T^\infty \frac{N(t,a)}{t^{u+1}}\,dt\ . \tag{4.3}$$

Now we proceed like in the proof of Borel's theorem as stated in [Nev1, p. 265] or in [Nev2, p. 261]: let a_1, a_2, a_3 be three different points of $J(f)$ and let $a \in J(f)$ be any point. Then it follows from FMT and SMT as stated above that, for every $t > T$,

$$N(t,a) - \Xi \le T(t) \le N(t,a_1) + N(t,a_2) + N(t,a_3) + S(t)\ . \tag{4.4}$$

Dividing this relation by t^{u+1} and integrating with respect to t gives

$$\int_T^\infty \frac{N(t,a)}{t^{u+1}}\,dt \le \sum_{j=1}^3 \int_T^\infty \frac{N(t,a_j)}{t^{u+1}}\,dt + A_u\ .$$

Here we used the fact that $S(r) = \mathcal{O}(\log(r))$, which implies that $\int_T^\infty \frac{S(t)}{t^{u+1}}\,dt = A_u < \infty$. Together with (4.3) we finally have

$$\Sigma(u,a) \le \Sigma(u,a_1) + \Sigma(u,a_2) + \Sigma(u,a_3) + u^2 A_u \tag{4.5}$$

for every $a \in J(f)$. $\square$

Remark 4.6. *If the order $\rho > 0$, then the above proof shows that $\Sigma(\rho,b) = \infty$ for all but at most two values $b \in \hat{\mathbb{C}}$ provided $\Sigma(\rho,a) = \infty$ for some $a \in \hat{\mathbb{C}}$. This property trivially also holds if $\rho = 0$. Note that Koebe's distortion theorem and hyperbolicity yield that the two exeptional values for this property cannot be in $J(f)$. Therefore $\Sigma(\rho,a) = \infty$ for all or none $a \in J(f)$ and this occurs if and only if*

$$\int \frac{T(r)}{r^\rho}\,dr = \infty\,.$$

5. Construction of conformal measures

Further properties of transfer operators $\mathcal{L}_t$ rely on the existence of conformal measures. Define now the topological pressure as follows.

$$\mathrm{P}(t) = \mathrm{P}(t,x) = \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{L}_t^n \mathbb{1}(x). \tag{5.1}$$

Note that because of hyperbolicity of the function f, Koebe's Distortion Theorem and density in $J(f)$ of the full backward orbit of any point in $J(f)$, the number $\mathrm{P}(t) = \mathrm{P}(t,x)$ is independent of $x \in J(f)$. We recall that m_t is called $e^{\mathrm{P}(t)}|f'|_\sigma^t$-conformal if $\frac{dm_t \circ f}{dm_t} = e^{\mathrm{P}(t)}|f'|_\sigma^t$ or, equivalently, if m_t is an eigenmeasure of the adjoint $\mathcal{L}_t^*$ of the transfer operator $\mathcal{L}_t$ with eigenvalue $e^{\mathrm{P}(t)}$. Note that then the measure m_t^e, the Euclidean version of m_t, defined by the requirement that $dm_t^e(z) = |z|^{\alpha_2 t} dm_t(z)$ is $e^{\mathrm{P}(t)}|f'|^t$-conformal. If $\mathrm{P}(t) = 0$, then these measures are called t-conformal. In [Su] Sullivan has proved that every rational function admits a probability conformal measure. As it is shown below, in the case of meromorphic functions the situation is not that far apart. All what you need for the existence of an $e^{\mathrm{P}(t)}|f'|_\sigma^t$-conformal

measure is the rapid derivative growth; no hyperbolicity is necessary. [1] We present here a very general construction of conformal measures.

Theorem 5.1. *If $f:\mathbb{C}\to\hat{\mathbb{C}}$ is a meromorphic function of finite order with non-empty Fatou set satisfying the rapid derivative growth condition, then for every $t>\rho/\alpha$ there exists a Borel probability $e^{\mathrm{P}(t)}|f'|_\sigma^t$-conformal measure m_t on $J(f)$.*

The rest of this section is devoted to the proof of Theorem 5.1. First of all, changing the system of coordinates by translation, we may assume without loss of generality that $0\notin J(f)$. Fix $x\in J(f)$. Observe that the transition parameter for the series

$$\Sigma_s=\sum_{n=1}^{\infty}e^{-ns}\mathcal{L}_t^n 1\!\!1(x)$$

is the topological pressure $\mathrm{P}(t)$. In other words, $\Sigma_s=+\infty$ for $s<\mathrm{P}(t)$ and $\Sigma_s<\infty$ for $s>\mathrm{P}(t)$. We assume that we are in the divergence case, e.g. $\Sigma_{\mathrm{P}(t)}=\infty$. For the convergence type situation the usual modifications have to be done (see [DU1] for details). For $s>\mathrm{P}(t)$, put

$$\nu_s=\frac{1}{\Sigma_s}\sum_{n=1}^{\infty}e^{-ns}(\mathcal{L}_t^n)^*\delta_x\ .$$

The following lemma follows immediately from definitions.

Lemma 5.2. *The following properties hold:*

(1) *For every $\varphi\in\mathcal{C}_b(\mathbb{C})$ we have*

$$\int\varphi d\nu_s=\frac{1}{\Sigma_s}\sum_{n=1}^{\infty}e^{-ns}\int\mathcal{L}_t^n\varphi d\delta_x=\frac{1}{\Sigma_s}\sum_{n=1}^{\infty}e^{-ns}\mathcal{L}_t^n\varphi(x)\ .$$

(2) ν_s *is a probability measure.*

(3) $$\frac{1}{e^s}\mathcal{L}_t^*\nu_s=\frac{1}{\Sigma_s}\sum_{n=1}^{\infty}e^{-(n+1)s}(\mathcal{L}_t^{n+1})^*\delta_x=\nu_s-\frac{1}{\Sigma_s}\frac{\mathcal{L}_t^*\delta_x}{e^s}\ .$$

The key ingredient of the proof of Theorem 5.1 is to show that the family $(\nu_s)_{s>P(t)}$ of Borel probability measures on $\mathbb{C}$ is tight and then to apply Prokhorov's Theorem. In order to accomplish this we put

$$U_R=\{z\in\mathbb{C}:|z|>R\}$$

and start with the following observation.

[1] If f is not hyperbolic then $\mathcal{F}_f=\emptyset$ may occur and our method does not work. But then the Lebesgue measure is 2-conformal.

Lemma 5.3. *For every $t > \rho/\alpha$ there is $C = C(t) > 0$ such that*

$$\mathcal{L}_t(\mathbb{1}_{U_R})(y) \le \frac{C}{R^{\alpha\gamma}} \ \text{ for every } \ y \in J(f),$$

where $\gamma = \frac{t-\rho/\alpha}{2}$.

Proof. From the rapid derivative growth condition (2.2) and Proposition 4.5, similarly as (4.1), we get for every $y \in J(f)$ that

$$\begin{aligned} \mathcal{L}_t(\mathbb{1}_{U_R})(y) &= \sum_{z\in f^{-1}(y)\cap U_R} |f'(z)|_\sigma^{-t} \le \kappa^t \sum_{z\in f^{-1}(y)\cap U_R} |z|^{-\alpha t} \\ &\le \frac{\kappa^t}{R^{\alpha\gamma}} \sum_{z\in f^{-1}(y)} |z|^{-(\rho+\alpha\gamma)} \le \frac{\kappa^t M_{\rho+\alpha\gamma}}{R^{\alpha\gamma}}. \end{aligned}$$

□

Now we are ready to prove the tightness we have already announced. We recall that this means that

$$\forall \varepsilon > 0 \ \exists R > 0 \ \text{ such that } \ \nu_s(U_R) \le \varepsilon \ \text{ for all } s > \mathrm{P}(t) \ .$$

Lemma 5.4. *The family $(\nu_s)_{s>\mathrm{P}(t)}$ of Borel probability measures on $\mathbb{C}$ is tight and, more precisely, there is $L > 0$ and $\delta > 0$ such that*

$$\nu_s(U_R) \le LR^{-\delta} \ \text{ for all } R > 0 \text{ and } s > \mathrm{P}(t) \ .$$

Proof. The first observation is that

$$\begin{aligned} \mathcal{L}_t^{n+1}(\mathbb{1}_{U_R})(x) &= \sum_{y\in f^{-n}(x)} \sum_{z\in f^{-1}(y)\cap U_R} \Big(|f'(z)|_\sigma \, |(f^n)'(y)|_\sigma\Big)^{-t} \\ &= \sum_{y\in f^{-n}(x)} |(f^n)'(y)|_\sigma^{-t} \mathcal{L}_t(\mathbb{1}_{U_R})(y) \le \frac{C}{R^{\alpha\gamma}} \mathcal{L}_t^n \mathbb{1}(x). \end{aligned}$$

where the last inequality follows from Lemma 5.3. Therefore, for every $s > \mathrm{P}(t)$, we get that

$$\begin{aligned} \nu_s(U_R) &= \frac{1}{\Sigma_s} \sum_{n=1}^{\infty} e^{-ns} \mathcal{L}_t^n(\mathbb{1}_{U_R})(x) \le \frac{C}{R^{\alpha\gamma}} \frac{1}{\Sigma_s} \sum_{n=1}^{\infty} e^{-ns} \mathcal{L}_t^{n-1} \mathbb{1}(x) \\ &= \frac{C}{R^{\alpha\gamma}} \frac{1}{e^s} \frac{1}{\Sigma_s} \Big(1 + \sum_{n=1}^{\infty} e^{-ns} \mathcal{L}_t^n \mathbb{1}(x)\Big) \le \frac{2C}{e^{P(t)}} \frac{1}{R^{\alpha\gamma}}. \end{aligned}$$

This shows Lemma 5.4 and the tightness of the family $(\nu_s)_{s>\mathrm{P}(t)}$. □

Now, choose a sequence $\{s_j\}_{j=1}^{\infty}$, $s_j > \mathrm{P}(t)$, converging down to $\mathrm{P}(t)$. In view of Prokhorov's Theorem and Lemma 5.4, passing to a subsequence, we may assume without loss of generality that the sequence $\{\nu_{s_j}\}_{j=1}^{\infty}$ converges weakly to a Borel

probability measure m_t on $J(f)$. It follows from Lemma 5.2 and the divergence property of Σ_s that $\mathcal{L}_t^* m_t = e^{P(t)} m_t$. The proof of Theorem 5.1 is complete.

6. Gibbs states

We now complete the proof of Theorem 1.1. The first observation is that one can have a better estimate than (4.1) in diminishing α_2 slightly. Suppose that the derivative of f satisfies the growth condition (2.2) with $\alpha_2' = \alpha_2 + \varepsilon$, $\varepsilon > 0$, instead of α_2. Then

$$|f'(z)|_\sigma = \frac{|z|^{\alpha_2}}{|f(z)|^{\alpha_2}} |f'(z)| \geq \frac{1}{\kappa} |z|^\alpha |f(z)|^\varepsilon \ , \quad z \in J(f) \, ,$$

which, along with Proposition 4.5, leads to the following important estimate of the transfer operator. For each $t > \rho/a$,

$$\mathcal{L}_t \mathbb{1}(w) \leq \frac{\kappa^t}{|w|^{t\varepsilon}} \sum_{z \in f^{-1}(w)} |z|^{-t\alpha} \leq \frac{\kappa^t M_{\alpha t}}{|w|^{t\varepsilon}} \quad \textit{for all} \;\; w \in J(f) \, . \tag{6.1}$$

An immediate advantage of this estimate is the following.

Lemma 6.1. *We have* $\lim_{w \to \infty} \mathcal{L}_t \mathbb{1}(w) = 0$.

The last ingredient we need in this section is the following straightforward consequence of Proposition 2.2, Koebe's Distortion Theorem, and the fact that $0 \notin J(f)$.

Lemma 6.2. *For every hyperbolic meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ satisfying the rapid derivative growth condition there exists a constant $K_\sigma \geq 1$, called σ-adjusted Koebe constant, such that if $R > 0$ is sufficiently small, then for every integer $n \geq 0$, every $w \in J(f)$, every $z \in f^{-n}(w)$ and all $x, y \in D_\sigma(w, R|w|^{-\alpha_2}) \cup D(w, R)$, we have that*

$$K_\sigma^{-1} \leq \frac{|(f_z^{-n})'(y)|_\sigma}{|(f_z^{-n})'(x)|_\sigma} \leq K_\sigma. \tag{6.2}$$

As an immediate consequence of this lemma and Montel's theorem, which implies that for every open set U intersecting the Julia set $J(f)$ and every point $z \in J(f)$ there exists $n \geq 0$ such that $U \cap f^{-n}(z) \neq \emptyset$, we conclude that the topological pressure

$$\mathrm{P}(t) = \lim_{n \to \infty} \frac{1}{n} \log \mathcal{L}_t^n(\mathbb{1})(w)$$

exists and is independent of $w \in J(f)$. From these two lemmas above and the existence of conformal measures (Theorem 5.1) one gets, following the arguments from formula (3.6) through Lemma 3.6 of [UZ2], the following uniform estimates for the normalized transfer operator

$$\hat{\mathcal{L}}_t = e^{-\mathrm{P}(t)} \mathcal{L}_t.$$

Proposition 6.3. *There exists $L > 0$ and, for every $R > 0$, there exists $l_R > 0$ such that*

$$l_R \le \hat{\mathcal{L}}_t^n \mathbb{1}(w) \le L$$

for all $n \ge 1$ and all $w \in J(f) \cap D(0,R)$.

This allows us to construct an everywhere positive, decreasing to zero at infinity, fixed point ψ of the normalized transfer operator $\hat{\mathcal{L}}_t$ by putting

$$\psi = \tilde{\psi}_t \Big/ \int \tilde{\psi}_t dm_t \quad \textit{with} \quad \tilde{\psi}_t(z) = \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \hat{\mathcal{L}}_t^k \mathbb{1}(z) \ , \ z \in J(f)$$

The Borel probability measure $\mu_t = \psi_t m_t$ is obviously f-invariant and equivalent to m_t. Repeating the appropriate reasonings from [UZ2] or [MyU], the proof of Theorem 1.1 follows.

7. Geometric applications

In the rest of the paper we derive several geometric consequences from the dynamical results proven in the previous sections. Our primary goal is to complete the proof of Theorem 1.3 (Bowen's formula). For this part we strengthen our assumptions and assume throughout the whole rest of the paper that f is dynamically regular.

Definition 7.1. *The meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ is called dynamically regular if it is hyperbolic, of positive and finite order ρ, satisfies the balanced derivative growth (condition (2.3)) and if it is of divergence type. In the case f is entire we assume instead of the divergence type assumption that, for any $A, B > 0$, there exists $R > 1$ such that*

$$\int_{\log R}^{R} \frac{T(r)}{r^{\rho+1}}\, dr - B\,(\log R)^{1-\rho} \ge A. \tag{7.1}$$

The divergence type assumption and also (7.1) do hold in particular if

$$\liminf_{r\to\infty} \frac{T(r)}{r^{\rho}} > 0.$$

This last condition is satisfied by all the examples given in Section 3.

In order to bring up geometric consequences, we need some information about the shape of the graph of the pressure function.

Proposition 7.2. *If $f : \mathbb{C} \to \hat{\mathbb{C}}$ is dynamically regular, then the following hold.*

(a) *The function $t \mapsto \mathrm{P}(t)$, $t > \rho/\alpha$, is convex and, consequently, continuous.*

(b) *The function* $t \mapsto \mathrm{P}(t)$, $t > \rho/\alpha$, *is strictly decreasing.*
(c) $\lim_{t\to+\infty} \mathrm{P}(t) = -\infty$.
(d) $\lim_{t\to(\rho/\alpha)^+} \mathrm{P}(t) > 0$.

Proof. Convexity of the pressure function $\mathrm{P}(t)$ follows immediately from its definition and Hölder's inequality. So, item (a) is proved. Items (b) and (c) are straightforward consequences of the expanding property.

It remains to show item (d). If f has a pole b of multiplicity q then, since f is balanced, $\alpha_2 = 1 + 1/q$ (see the discussion on elliptic functions in Section 3). The result in [My2] shows then that

$$\mathrm{P}(\rho/\alpha) \geq 0$$

with strict inequality if f is of divergence type (see Remark 3.2 of [My2]).

So, let finally f be entire. Notice first that, with the balanced growth condition, the calculations leading to (4.1) give the following lower estimate.

$$\mathcal{L}_t 1\!\!1(w) = \sum_{z\in f^{-1}(w)} |f'(z)|_\sigma^{-t} \geq \kappa^{-t} \sum_{z\in f^{-1}(w)} |z|^{-\alpha t} \quad , \; w \in J(f), \tag{7.2}$$

for all $t > \rho/\alpha$. Denote now for every $R > 0$

$$\Sigma^R(u,a) = \sum_{z\in f^{-1}(a)\cap D(0,R)} |z|^{-u} \quad , \; a \in J(f) \textit{ and } u \geq \rho.$$

In order to proof $\mathrm{P}(\rho/\alpha) > 0$ it suffices to show that for a given $A > 0$ there exists $R > T$ such that

$$\Sigma^R(\rho, a) \geq A \quad \textit{for all} \;\; a \in J(f) \cap D(0,R).$$

Let $R > T$ and let $a \in D(0,R) \cap J(f)$. We get precisely in the same way as in (4.3) that

$$\Sigma^R(\rho,a) \geq \rho^2 \int_0^R \frac{N(t,a)}{t^{\rho+1}}\, dt \geq \int_{\log|a|}^R \frac{N(t,a)}{t^{\rho+1}}\, dt.$$

From the sharp form of the SMT (Lemma 4.4), it follows that

$$\begin{aligned} \Sigma^R(\rho,a) &\geq \rho^2 \int_{\log|a|-\Delta}^{R-\Delta} \frac{N(r+\Delta,a)}{r^{\rho+1}} \left(\frac{r}{r+\Delta}\right)^{\rho+1} dr \asymp \int_{\log|a|-\Delta}^{R-\Delta} \frac{4N(r+\Delta,a)}{r^{\rho+1}} dr \\ &\geq \int_{\log|a|-\Delta}^{R-\Delta} \frac{T(r)}{r^{\rho+1}} dr - C_1 - C_2 \left(\log|a|\right)^{1-\rho} \end{aligned}$$

for some constants $C_1, C_2 > 0$. If the order $\rho \geq 1$ then $(\log|a|)^{1-\rho}$ is bounded above. Consequently there are $C_3, C_4 > 0$ such that

$$\Sigma^R(\rho,a) \geq C_3 \int_{\log R-\Delta}^{R-\Delta} \frac{T(r)}{r^{\rho+1}} dr - C_4.$$

In the case when $0<\rho<1$, we have $(\log|a|)^{1-\rho}\le(\log R)^{1-\rho}\asymp(\log(R-\Delta))^{1-\rho}$. Therefore

$$\Sigma^R(\rho,a)\ge C_3\int_{\log R-\Delta}^{R-\Delta}\frac{T(r)}{r^{\rho+1}}dr-C_1-C_5\left(\log(R-\Delta)\right)^{1-\rho}$$

for some $C_5>0$. The assertion follows now from the assumption (7.1). □

A direct application of Theorem 1.1 gives now

Corollary 7.3. *If $f:\mathbb{C}\to\hat{\mathbb{C}}$ is dynamically regular, then there exists a unique $h>\rho/\alpha$ such that $\mathrm{P}(h)=0$ and f has a $|f'|_\sigma^h$-conformal measure m_h.*

The definitions of Hausdorff measure as well as Hausdorff dimension can be found for example in [Mat] or [PU]. The symbol H_σ^t refers to the t-dimensional Hausdorff measure evaluated with respect to the Riemannian metric $d\sigma$. Fix $t>\rho/\alpha$. By Theorem 5.1 there exists m_t, an $e^{\mathrm{P}(t)}|f'|_\sigma^t$-conformal measure, and let m_t^e be its Euclidean version defined in the previous section. Then a straightforward calculation shows that

$$\frac{dm_t^e\circ f}{dm_t^e}(z)=e^{\mathrm{P}(t)}|f'(z)|^t,\ z\in J(f). \tag{7.3}$$

Fix any radius

$$R\in(0,\delta(f)).$$

So, if $z\in J(f)$, $n\ge0$, and $z\in f^{-n}(w)$, then there exists a unique holomorphic inverse branch $f_z^{-n}:D(w,4R)\to\mathbb{C}$ of f^n sending w to z. Recall that K_σ is the σ-adjusted Koebe constant produced in Lemma 6.2. It follows from this lemma that

$$D_\sigma\big(z,K_\sigma^{-1}R|w|^{-\alpha_2}|(f^n)'(z)|_\sigma^{-1}\big)\subset f_z^{-n}\big(D_\sigma(w,R|w|^{-\alpha_2})\big)\subset D_\sigma\big(z,K_\sigma R|w|^{-\alpha_2}|(f^n)'(z)|_\sigma^{-1}\big) \tag{7.4}$$

and that

$$m_t\big(f_z^{-n}\big(D_\sigma(w,R|w|^{-\alpha_2})\big)\big)\asymp e^{-\mathrm{P}(t)n}|(f^n)'(z)|_\sigma^{-t}m_t\big(D_\sigma(w,R|w|^{-\alpha_2})\big). \tag{7.5}$$

We recall that the radial Julia set is the set of points of $J(f)$ that do not escape to infinity:

$$J_r(f)=\{z\in J(f):\ \liminf_{n\to\infty}|f^n(z)|<\infty\}$$

and, obviously,

$$J_r(f)=\bigcup_{M>0}J_{r,M}(f)=\bigcup_{M>0}\{z\in J(f):\liminf_{n\to\infty}|f^n(z)|<M\}.$$

8. Proof of Bowen's formula

We start the proof of Bowen's formula (Theorem 1.3) by the following observation which, together with Lemma 7.3, shows in particular that $\mathrm{HD}(J_r(f)) \leq h$.

Lemma 8.1. *If $t > \rho/\alpha$ such that $\mathrm{P}(t) \leq 0$, then $\mathrm{H}^t_\sigma(J_r(f)) < +\infty$.*

Proof. Since μ_t is an ergodic measure there is $M > 0$ so large that $\mu_t(J_{r,M}(f)) = 1$. Consequently $m_t(J_{r,M}(f)) = 1$. Since $J(f) \cap \overline{D}(0,M)$ is a compact set,

$$Q_M := \inf\{m_t\big(D_\sigma(w, R|w|^{-\alpha_2}) : w \in J(f) \cap D(0,M)\big)\} > 0.$$

Now, fix $z \in J_{r,M}(f)$ and consider an arbitrary integer $n \geq 0$ such that $f^n(z) \in D(0,M)$. Recall that $D(0,T) \cap J(f) = \emptyset$. It follows from (7.4) and (7.5) that

$$\begin{aligned} m_t\big(D_\sigma\big(z, K_\sigma R|f^n(z)|^{-\alpha_2}|(f^n)'(z)|_\sigma^{-1}\big)\big) &\succeq \\ &\succeq e^{-\mathrm{P}(t)n}|(f^n)'(z)|_\sigma^{-t} m_t\big(D_\sigma(f^n(z), R|f^n(z)|^{-\alpha_2})\big) \\ &\geq Q_M(K_\sigma R)^{-t}e^{-\mathrm{P}(t)n}|f^n(z)|^{\alpha_2 t}\big(K_\sigma R|f^n(z)|^{-\alpha_2}|(f^n)'(z)|_\sigma^{-1}\big)^t \\ &\geq Q_M(K_\sigma R)^{-t}T^{\alpha_2 t}\big(K_\sigma R|f^n(z)|^{-\alpha_2}|(f^n)'(z)|_\sigma^{-1}\big)^t. \end{aligned}$$

Thus, there exists $c > 0$ such that for every $z \in J_{r,M}(f)$

$$\limsup_{r\to 0}\frac{m_t(D_\sigma(z,r))}{r^t} \geq \limsup_{n\to\infty}\frac{m_t\big(D_\sigma\big(z, K_\sigma R|f^n(z)|^{-\alpha_2}|(f^n)'(z)|_\sigma^{-1}\big)\big)}{\big(K_\sigma R|f^n(z)|^{-\alpha_2}|(f^n)'(z)|_\sigma^{-1}\big)^t} \geq c.$$

Applying now Besicovic's Covering Theorem, it immediately follows from this inequality that $\mathrm{H}^t_\sigma(J_{r,M}(f)) \leq c^{-1}$. Since for every $x \geq M$, $m_t\big(J_{r,x+1}(f) \setminus J_{r,x}(f)\big) = 0$, an argument similar to the one above gives that $\mathrm{H}^t_\sigma\big(J_{r,x+1}(f) \setminus J_{r,x}(f)\big) = 0$. Since $J_r(f) = J_{r,M}(f) \cup \bigcup_{n=0}^\infty\big(J_{r,M+n+1}(f) \setminus J_{r,M+n}(f)\big)$, the proof is complete. □

In order to complete the proof of Bowen's formula we have to establish that $\mathrm{HD}(J_r(f)) \geq h$. We will do this in adapting the corresponding proof in [UZ2]. The first step is to show that f has a finite and strictly positive Lyapunov exponent.

Lemma 8.2. *We have that*

$$0 < \chi = \int \log|f'|d\mu_h = \int \log|f'|_\sigma d\mu_h < \infty\ .$$

Proof. The equality $\int \log|f'|d\mu_h = \int \log|f'|_\sigma d\mu_h$ follows from

$$\log|f'|_\sigma(z) = \log|f'(z)| + \alpha_2(\log|z| - \log|f(z)|)$$

and the $f-$invariance of μ_h. We have to prove finiteness of $\int \log f'_\sigma d\mu_h$. In order to do so, consider the annulus $A_j = D(0,2^{j+1}) \setminus D(0,2^j)$. In this annulus we have

(i) $\mu_h(A_j) \preceq 2^{-j\delta}$ because of Lemma 5.4 and the fact that $d\mu_h = \psi_h dm_h$ with ψ_h bounded.

(ii) $f'_\sigma(z) \preceq |z|^\alpha \preceq 2^{j\alpha}$ due to the balanced growth condition (2.3).

The finiteness of the integrals in the lemma follows. Finally $\chi > 0$ since f is expanding. $\square$

We can now complete the proof of Theorem 1.3 by establishing the following.

Lemma 8.3. $\mathrm{HD}(J_r(f)) \geq h$.

Proof. Fix $\varepsilon >$ such that the Lyapunov exponent defined in Lemma 8.2 $\chi > \varepsilon$. Since $\mu_h(J_r(f)) = 1$ and since μ_h is ergodic $f-$invariant, it follows from Birkhoff's ergodic theorem and Jegorov's theorem that there exists a Borel set $Y \subset J_r(f)$ and an integer $K \geq 1$ such that $\mu_h(Y) \geq \frac{1}{2}$ and such that for every $z \in Y$ and $n \geq k$

$$\left|\frac{1}{n}\log|(f^n)'(z)| - \chi\right| < \varepsilon \ \ \textit{and} \ \ \left|\frac{1}{n}\log|(f^n)'(z)|_\sigma \ - \chi\right| < \varepsilon. \tag{8.1}$$

Let $R = \mathrm{dist}(\mathcal{P}_f, J(f))/4$. Given $z \in Y$ and $r \in (0, R)$, let $n \geq 0$ be the largest integer such that

$$D(z,r) \subset f_z^{-n}(D(f^n(z), R)).$$

There is $r_z > 0$ such that for any $0 < r < r_z$ the integer n defined above is $n \geq k$. By the definition of n, $D(z,r)$ is not contained in $f_z^{-(n+1)}(D(f^{n+1}(z), R))$. Koebe's distortion theorem yields now

$$r \leq KR|(f^n)'(z)|^{-1} \quad \textit{and} \quad r \geq K^{-1}R|(f^{n+1})'(z)|^{-1}. \tag{8.2}$$

Passing to the h-conformal measure m_h, we get from (8.1) that

$$\begin{aligned} m_h(D(z,r)) &\leq m_h\big(f_z^{-n}(D(f^n(z), R)\big) \asymp |(f^n)'(z)|_\sigma^{-h} m_h\big(D(f^n(z), \delta)\big) \\ &\leq |(f^n)'(z)|_\sigma^{-h} \leq e^{-hn(\chi-\varepsilon)}. \end{aligned}$$

On the other hand, (8.1) together with (8.2) give

$$e^{-(n+1)(\chi+\varepsilon)} \leq |(f^{n+1})'(z)|^{-1} \preceq r.$$

Therefore

$$m_h(D(z,r)) \preceq r^{h\left(\frac{n+1}{n}\frac{\chi-\varepsilon}{\chi+\varepsilon}\right)}.$$

When $r \to 0$ then $n = n(r) \to \infty$ from which we get that

$$\limsup_{r\to 0} \frac{m_h(D(z,r))}{r^{h-\varepsilon'}} \preceq 1$$

for every $\varepsilon' > 0$. This gives $\mathrm{HD}(J_r(f)) \geq h - \varepsilon'$ and the lemma follows in taking $\varepsilon' \to 0$. $\square$

9. Real analyticity of the hyperbolic dimension

In this section we prove Theorem 1.7. From now on we suppose $\alpha_1 \geq 0$.

9.1. **J-stability.** The work of Lyubich and Mañé-Sad-Sullivan [L1, MSS] on the structural stability of rational maps has been generalized to entire functions of the Speiser class by Eremenko-Lyubich [EL]. Note also that they show that any entire function of the Speiser class is naturally imbedded in a holomorphic family of functions in which the singular points are local parameters.

Here we collect and adapt to the meromorphic setting the facts that are important for our needs. We also deduce from the bounded deformation assumption of $\mathcal{M}_\Lambda$ near f_{λ^0} a bounded speed condition of the involved holomorphic motions. A *holomorphic motion* of a set $A \subset \mathbb{C}$ over U originating at λ^0 is a map $h : U \times A \to \mathbb{C}$ satisfying the following conditions:

(1) The map $\lambda \mapsto h(\lambda, z)$ is holomorphic for every $z \in A$.
(2) The map $h_\lambda : z \mapsto h_\lambda(z) = h(\lambda, z)$ is injective for every $\lambda \in U$.
(3) $h_{\lambda^0} = id$.

The λ–lemma [MSS] asserts that such a holomorphic motion extends in a quasiconformal way to the closure of A. Further improvements, resulting in the final version of Slodkowski [Sk], show that each map h_λ is the restriction of a global quasiconformal map of the sphere $\hat{\mathbb{C}}$. Let us call $f_{\lambda^0} \in \mathcal{M}_\Lambda$ *holomorphically J-stable* if there is a neighborhood $U \subset \Lambda$ of λ^0 and a holomorphic motion h_λ of $\mathcal{J}(f_{\lambda^0})$ over U such that $h_\lambda(\mathcal{J}(f_{\lambda^0})) = \mathcal{J}(f_\lambda)$ and

$$h_\lambda \circ f_{\lambda^0} = f_\lambda \circ h_\lambda \quad on \;\; \mathcal{J}(f_{\lambda^0})$$

for every $\lambda \in U$.

Lemma 9.1. *A function $f_{\lambda^0} \in \mathcal{M}_\Lambda$ is holomorphically J-stable if and only if, for every singular value $a_{j,\lambda^0} \in sing(f_{\lambda^0}^{-1})$, the family of functions*

$$\lambda \mapsto f_\lambda^n(a_{j,\lambda}) \; , \;\; n \geq 1,$$

is normal in a neighborhood of λ^0.

Proof. This can be proved precisely like for rational functions because the functions in the Speiser class $\mathcal{S}$ do not have wandering nor Baker domains (see [L2] or [BM, p. 102]). □

From this criterion together with the description of the components of the Fatou set one easily deduces the following.

Lemma 9.2. *Each $f_{\lambda^0} \in \mathcal{HM}_\Lambda$ is holomorphically J–stable and $\mathcal{HM}_\Lambda$ is open in $\mathcal{M}_\Lambda$.*

We now investigate the speed of the associated holomorphic motion.

Proposition 9.3. *Let $f_{\lambda^0} \in \mathcal{HM}_\Lambda$ and let h_λ be the associated holomorphic motion over $U \subset \Lambda$ (cf. Lemma 9.2). If $\mathcal{M}_U$ is of bounded deformation, then there is $C > 0$ such that*

$$\left| \frac{\partial h_\lambda(z)}{\partial \lambda_j} \right| \leq C$$

for every $z \in \mathcal{J}(f_{\lambda^0})$ and $j = 1, ..., N$. It follows that h_λ converges to the identity map uniformly on $\mathcal{J}(f_{\lambda^0})$ and, replacing U by a smaller neighborhood if necessary, that there exists $0 < \tau \le 1$ such that h_λ is τ-Hölder for every $\lambda \in U$.

Proof. Let h_λ be the holomorphic motion such that $f_\lambda \circ h_\lambda = h_\lambda \circ f_{\lambda^0}$ on $\mathcal{J}(f_{\lambda^0})$ for $\lambda \in U$ and such that there are $c > 0$ and $\rho > 1$ for which

$$|(f_\lambda^n)'(z)| \ge c\rho^n \quad \textit{for every } n \ge 1,\ z \in \mathcal{J}_{f_\lambda} \textit{ and } \lambda \in U. \tag{9.1}$$

(cf. Fact 2.3; this is the only place where $\alpha_1 \ge 0$ is used). Denote $z_\lambda = h_\lambda(z)$ and consider

$$F_n(\lambda, z) = f_\lambda^n(z_\lambda) - z_\lambda.$$

The derivative of this function with respect to λ_j gives

$$\frac{\partial}{\partial \lambda_j} F_n(\lambda, z) = \frac{\partial f_\lambda^n}{\partial \lambda_j}(h_\lambda(z)) + (f_\lambda^n)'(h_\lambda(z)) \frac{\partial}{\partial \lambda_j} h_\lambda(z) - \frac{\partial}{\partial \lambda_j} h_\lambda(z).$$

Suppose that z is a repelling periodic point of period n. Then $\lambda \mapsto F_n(\lambda, z) \equiv 0$ and it follows from (9.1) that

$$\left| \frac{\partial h_\lambda(z)}{\partial \lambda_j} \right| = \left| \frac{\frac{\partial f_\lambda^n}{\partial \lambda_j}(z_\lambda)}{1 - (f_\lambda^n)'(z_\lambda)} \right| \preceq \left| \frac{\frac{\partial f_\lambda^n}{\partial \lambda_j}(z_\lambda)}{(f_\lambda^n)'(z_\lambda)} \right| = \Delta_{n,j}.$$

Since $\frac{\partial f_\lambda^n}{\partial \lambda_j}(z_\lambda) = \frac{\partial f_\lambda}{\partial \lambda_j}\big(f_\lambda^{n-1}(z_\lambda)\big) + f_\lambda'\big(f_\lambda^{n-1}(z_\lambda)\big) \frac{\partial f_\lambda^{n-1}}{\partial \lambda_j}(z_\lambda)$ we have

$$\Delta_{n,j} \le \frac{\left| \frac{\partial f_\lambda}{\partial \lambda_j}(f_\lambda^{n-1}(z_\lambda)) \right|}{\left| f_\lambda'(f_\lambda^{n-1}(z_\lambda)) \right|} \frac{1}{\left| (f_\lambda^{n-1})'(z_\lambda) \right|} + \Delta_{n-1,j}.$$

Making use of the expanding (9.1) and the bounded deformation (1.4) properties it follows that

$$\Delta_{n,j} \le \frac{M}{c\rho^{n-1}} + \Delta_{n-1,j}.$$

The conclusion comes now from the density of the repelling cycles in the Julia set $\mathcal{J}(f_{\lambda^0})$:

$$\left| \frac{\partial h_\lambda(z)}{\partial \lambda_j} \right| \preceq \frac{M}{c} \frac{\rho}{\rho - 1} \quad \textit{for every } z \in \mathcal{J}(f_{\lambda^0}).$$

The Hölder continuity property is now standard (see [UZ2]). □

Concerning the divergence type condition and the growth condition on the characteristic function (7.1), these are stable in the sense that if f_{λ_0} satisfies it, then f_λ also has this property for all λ in some neighbourhood of λ_0. For example, if f_{λ_0} is entire, then

$$T(r, f_{\lambda_0}) = m(r, \infty) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f_{\lambda_0}(re^{i\theta})| \, d\theta.$$

If (7.1) holds for this function, then it follows from that expression and from the uniform convergence on compact sets of f_λ to f_{λ_0} that (7.1) does hold for all f_λ, λ in some neighborhood of λ_0.

9.2. **The spectral gap of the (real) transfer operator.** In order to get the necessary spectral properties of the transfer operator, one does work with the space of Hölder continuous functions $\mathcal{H}_\tau = \mathcal{H}_\tau(J(f), \mathbb{C})$, $0 < \tau \leq 1$. However, the function $|f'|_\sigma^{-1}$ is not necessary in this space. It follows from the distortion property (9.2) below that it belongs to the following slightly more general one. In order to introduce it consider $w \in J(f)$ and denote the τ–variation of a function $g : J(f) \cap D(w, \delta) \to \mathbb{C}$ by

$$v_{\tau,w}(g) = \sup\left\{ \frac{|g(x) - g(y)|}{|x-y|^\tau} \; ; \;\; x, y \in J(f) \cap D(w, \delta) \right\}.$$

The Hölder space H_τ we work with consists in bounded functions $g : J(f) \to \mathbb{C}$ such that $v_{\tau,w}(g \circ f_a^{-1})$ is bounded uniformly in $w \in J(f)$ and $a \in f^{-1}(w)$. Denote

$$v_\tau(g) = \sup_{w \in J(f)} \sup_{a \in f^{-1}(w)} v_{\tau,w}(g \circ f_a^{-1}).$$

The space H_τ endowed with the norm $\|g\|_\tau = v_\tau(g) + \|g\|_\infty$ is a Banach space densely contained in C_b. Here is the classical estimation which is based on the Hölder property of $g \in H_\tau$ and the expanding property of f:

$$\begin{aligned}
\left|\hat{\mathcal{L}}_t^n g(z) - \hat{\mathcal{L}}_t^n g(w)\right| &= e^{-nP(t)} \Big| \sum_{a \in f^{-n}(z)} |(f^n)'(a)|_\sigma^{-t} g(a) - \sum_{b \in f^{-n}(w)} |(f^n)'(b)|_\sigma^{-t} g(b) \Big| \\
&\leq e^{-nP(t)} \sum_{a \in f^{-n}(z)} |(f^n)'(a)|_\sigma^{-t} \left| g(f_a^{-n}(z)) - g(f_a^{-n}(w)) \right| \\
&+ e^{-nP(t)} \sum_{b \in f^{-n}(w)} \left| |(f^n)'(f_b^{-n}(z))|_\sigma^{-t} - |(f^n)'(b)|_\sigma^{-t} \right| \left| g(b) \right| \\
&= I + II
\end{aligned}$$

for $z, w \in J(f)$ with $|z - w| < \delta = \delta(f)$ and where f_a^{-n} is the inverse branch of f^{-n} defined on $D(z, \delta)$ such that $f_a^{-n}(z) = a$. The majorization of the first term goes as follows:

$$\begin{aligned}
I &\leq v_\tau(g) e^{-nP(t)} \sum_{a \in f^{-n}(z)} |(f^n)'(a)|_\sigma^{-t} \left| f_{f(a)}^{-(n-1)}(z) - f_{f(a)}^{-(n-1)}(w) \right|^\tau \\
&\leq v_\tau(g) \|\hat{\mathcal{L}}_t^n\|_\infty \sup_{a \in f^{-n}(z)} |f_{f(a)}^{-(n-1)}(z) - f_{f(a)}^{-(n-1)}(w)|^\tau \preceq v_\tau(g) \rho^{-(n-1)\tau} |z - w|^\tau.
\end{aligned}$$

Concerning the second part, one has to observe that Koebe's distortion theorem implies that for any $n \geq 1$ and $z, w, \in J(f)$ with $|z - w| < \delta(f)$

$$\left| |(f^n)'(f_a^{-n}(z))|_\sigma^{-t} - |(f^n)'(f_a^{-n}(w))|_\sigma^{-t} \right| \preceq |(f^n)'(f_a^{-n}(z))|_\sigma^{-t} |z - w| \tag{9.2}$$

where $a \in f^{-n}(z)$. Therefore

$$II \preceq e^{-nP(t)} \sum_{b \in f^{-n}(w)} |(f^n)'(b)|_\sigma^{-t} |z-w||g(b)| \preceq \|\hat{\mathcal{L}}_t^n\| \|g\|_\infty |z-w|.$$

Altogether we have

$$\left|\hat{\mathcal{L}}_t^n g(z) - \hat{\mathcal{L}}_t^n g(w)\right| \preceq \left(\rho^{-(n-1)\tau} v_\tau(g) + \|g\|_\infty\right)|z-w|^\tau \tag{9.3}$$

for all $z, w \in J(f)$ with $|z-w| < \delta(f)$. We proved

Lemma 9.4. *$\hat{\mathcal{L}}_t(H_\tau) \subset H_\tau$ and, for any $g \in H_\tau$ and $n \geq 1$,*

$$\|\hat{\mathcal{L}}_t^n(g)\|_\tau \preceq \rho^{-(n-1)\tau} v_\tau(g) + \|g\|_\infty.$$

If B is a bounded subset of H_τ then (9.3) and the fact that $\|\hat{\mathcal{L}}_t^n\|_\infty$ is uniformly bounded yields that $\mathcal{F} = \{\hat{\mathcal{L}}_t(g);\ g \in B\}$ is a equicontinuous bounded subfamily of $(C_b, \|.\|_\infty)$. The following observation follows then precisely like in [UZ2, Lemma 4.2] (using $\lim_{w\to\infty} \mathcal{L}_t 1\!\!1(w) = 0$ which is Lemma 6.1).

Lemma 9.5. *If B is a bounded subset of H_τ, then $\hat{\mathcal{L}}_t(B)$ is a precompact subset of $(C_b, \|.\|_\infty)$.*

We are now in the position to apply Ionescu-Tulcea and Marinescu's Theorem 1.5 in [IM]. Combined with [DU2] (see [UZ2] were these facts are explained in detail) we finally get:

Proposition 9.6. *For all $t > \rho/\alpha$ there is $r \in (0,1)$ such that the spectrum $\sigma(\hat{\mathcal{L}}_t) \subset \mathbb{D}(0,r) \cup \{1\}$ and the number 1 is a simple isolated eigenvalue of the operator $\hat{\mathcal{L}}_t$ of H_τ.*

9.3. **Complexified transfer operator.** In the remainder of the paper we consider a hyperbolic function $f_{\lambda^0} \in \mathcal{HM}_\Lambda$. Let $U \subset \Lambda$ be a neighborhood of λ_0 on which f_λ is hyperbolic and holomorphically J–stable and let

$$\mathcal{L}_{t,\lambda} g(w) = \sum_{z \in f_\lambda^{-1}(w)} |f_\lambda'(z)|_\sigma^{-t} g(z)\ , \quad t > \frac{\rho}{\alpha},$$

be the induced family of (real) transfer operators acting continuously on $C_b(\mathcal{J}(f_\lambda), \mathbb{C})$ and on $H_1(\mathcal{J}(f_\lambda), \mathbb{C})$. In order to be able to work on the fixed Julia set $\mathcal{J}(f_{\lambda^0})$ we conjugate these operators by $T_\lambda : C_b(\mathcal{J}(f_\lambda), \mathbb{C}) \to C_b(\mathcal{J}(f_{\lambda^0}), \mathbb{C})$ where $T_\lambda(g) = g \circ h_\lambda$ and where h_λ is the associated holomorphic motion. Put

$$L(t,\lambda) = T_\lambda \circ \mathcal{L}_{t,\lambda} \circ T_\lambda^{-1}$$

to be the resulting bounded operator of $C_b = C_b(\mathcal{J}(f_{\lambda^0}), \mathbb{C})$. We have that

$$L(t,\lambda)(g)(w) = \sum_{z\in f_{\lambda^0}^{-1}(w)} |f_\lambda'(h_\lambda(z))|_\sigma^{-t} g(z)\ ,\quad w\in \mathcal{J}(f_{\lambda^0})\ ,\quad g\in C_b.$$

Our aim is to establish real analyticity of the hyperbolic dimension of f_λ. In order to do so we have to embed these operators in a holomorphic family

$$(t,\lambda)\in \mathbb{C}\times\mathbb{C}^{2d} \to L(t,\lambda)\in L(H_\tau).$$

In order to do so, we follow [UZ2] and start with complexifying the potentials $|f_\lambda'|_\sigma^{-t}\circ h_\lambda$. Denote again $z_\lambda = h_\lambda(z)$, $z\in\mathcal{J}(f_{\lambda^0})$ and $\lambda\in\mathbb{D}_{\mathbb{C}^d}(\lambda^0,R)$. Remember that $h_\lambda\to id$ uniformly in $\mathcal{J}(f_{\lambda^0})$ (Proposition 9.3). Since $0\notin\mathcal{J}(f_{\lambda^0})$ the function

$$\Psi_z(\lambda) = \frac{f_\lambda'(z_\lambda)}{f_{\lambda^0}'(z)}\left(\frac{z_\lambda}{z}\right)^{\alpha_2}\left(\frac{f_{\lambda^0}(z)}{f_\lambda(z_\lambda)}\right)^{\alpha_2}$$

is well defined on the simply connected domain $\mathbb{D}_{\mathbb{C}^d}(\lambda^0,R)$. Here we choose $w\mapsto w^{\alpha_2}$ so that this map fixes 1 which implies that

$$\Psi_z(\lambda^0)=1 \quad \textit{for every}\ \ z\in\mathcal{J}_0=\mathcal{J}(f_{\lambda^0})\setminus f_{\lambda^0}^{-1}(\infty).$$

For this function one has the following uniform estimate.

Lemma 9.7. *For every $\varepsilon>0$ there is $0<r_\varepsilon<R$ such that $|\Psi_z(\lambda)-1|<\varepsilon$ for every $\lambda\in\mathbb{D}_{\mathbb{C}^d}(\lambda^0,r_\varepsilon)$ and every $z\in\mathcal{J}_0$.*

Proof. Suppose to the contrary that there is $\varepsilon>0$ such that for some $r_j\to 0$ there exists $\lambda_j\in\mathbb{D}_{\mathbb{C}^d}(\lambda^0,r_j)$ and $z_j\in\mathcal{J}_0$ with $|\Psi_{z_j}(\lambda_j)-1|>\varepsilon$. Then the family of functions

$$\mathcal{F}=\{\Psi_z\,;z\in\mathcal{J}_0\}$$

cannot be normal on any domain $\mathbb{D}_{\mathbb{C}^d}(\lambda^0,r)$, $0<r<R$. This is however not true. Indeed, the balanced growth condition (2.3) yields

$$|\Psi_z(\lambda)|\le\kappa^2\left|\frac{z_\lambda}{z}\right|^{\alpha}\qquad \textit{for every}\ z\in\mathcal{J}_0\ \textit{and}\ |\lambda-\lambda^0|<R.$$

Since $h_\lambda\to Id$ uniformly in $\mathbb{C}$ it follows immediately that $\mathcal{F}$ is normal on some disk $\mathbb{D}_{\mathbb{C}^d}(\lambda^0,r)$, $0<r<R$. □

We can now proceed precisely as in [UZ2] (or in [CS2]). Embed

$$\lambda=(\lambda_{d-1},...,\lambda_0)=(x_{d-1}+iy_{d-1},...,x_0+iy_0)\in\mathbb{C}^d$$

into $\mathbb{C}^{2d}$ by the formula $\lambda\mapsto(x_{d-1},y_{d-1},...,x_0,y_0)\in\mathbb{C}^{2d}$, replace in the power series of the, for every $z\in\mathcal{J}_0$, real analytic functions

$$(t,\lambda)\mapsto|f_\lambda'(z_\lambda)|_\sigma^{-t}=\exp\{-t\Re\log f_{\lambda,\sigma}'(z_\lambda)\}=|f_{\lambda^0}'(z)|_\sigma^{-t}\exp\{-t\Re\log\Psi_z(\lambda)\},$$

$\|\lambda-\lambda_0\|<R$ and $\Re(t)>\frac{\rho}{\alpha}$, the real numbers $x_j=\Re\lambda_j$, $y_j=\Im\lambda_j$ by complex numbers and obtain by a straightforward adaption of the arguments given in [UZ2, CS2] the following:

Proposition 9.8. *There is $R > 0$ such that, for every $z \in \mathcal{J}_0$, the function*

$$(t, \lambda) \mapsto \varphi_{t,\lambda}(z) = |f'_{\lambda^0}(z)|_\sigma^{-t} \exp\{-t\Re \log \Psi_z(\lambda)\}$$

can be extended to a holomorphic function on $\{\Re t > \frac{\rho}{\alpha}\} \times \mathbb{D}_{\mathbb{C}^{2d}}(\lambda_0, R)$. In addition, this extension that we still denote $\varphi_{t,\lambda}$ has the following properties:

1. $|\varphi_{t,\lambda}(z)| \asymp |f'_{\lambda^0}(z)|_\sigma^{-t}$.
2. *There is $0 < \tau \le 1$ such that $\varphi_{t,\lambda} \in H_\tau$ and $(t, \lambda) \mapsto \varphi_{t,\lambda} \in H_\tau$ is continuous.*
3. *$\varphi_{t,\lambda}$ is uniformly dynamically Hölder.*

A continuous function $\varphi : \mathcal{J}(f_{\lambda^0}) \to \mathbb{C}$ is called c_φ*–dynamically Hölder* of exponent τ if

$$|\varphi_n((f_{\lambda^0})_a^{-n}(z)) - \varphi_n((f_{\lambda^0})_a^{-n}(w))| \le c_\varphi |\varphi_n((f_{\lambda^0})_a^{-n}(z))||z - w|^\tau$$

for $a \in f_{\lambda^0}^{-n}(z)$, $|z - w| < \delta(f_{\lambda^0})$ and with $\varphi_n(a) = \varphi(a)\varphi(f_{\lambda^0}(a)) \cdot ... \cdot \varphi(f_{\lambda^0}^{n-1}(a))$. As we noted in (9.2), $\varphi_{t,\lambda^0}(z) = |f'_{\lambda^0}(z)|_\sigma^{-t}$ is dynamically Hölder. The family of potentials $\varphi_{t,\lambda}$ is called *uniformly dynamically Hölder* if the involved constants τ, c_φ above can be chosen to be valid for all the potentials of the family. Item (1) of the preceding proposition means in particular that the transfer operators

$$L(t, \lambda)(g)(w) = \sum_{z \in f_{\lambda^0}^{-1}(w)} \varphi_{t,\lambda}(z) g(z) \tag{9.4}$$

are (uniformly) bounded on C_b (such potentials are also called (uniformly) summable). In fact, much more is true since Proposition 9.8 together with Corollary 7.7 of [UZ2] yield:

Corollary 9.9. *There are $0 < \tau \le 1$ and $R > 0$ such that the operators $L(t, \lambda)$ are bounded operators of H_τ and such that the map*

$$(t, \lambda) \in \{\Re t > \rho/\alpha\} \times \mathbb{D}_{\mathbb{C}^{2d}}(\lambda_0, R) \mapsto L(t, \lambda) \in L(H_\tau)$$

is holomorphic.

9.4. **Real analyticity of the hyperbolic dimension.** We are now in position to proof Theorem 1.7. We take the notation of the preceding section, in particular $f_{\lambda^0} \in \mathcal{H}$ is a hyperbolic function. Consider a real $t_0 > \frac{\rho}{\alpha}$. Then we have $\mathcal{L}_{t_0,\lambda_0} = L(t_0, \lambda_0) \in L(H_\tau)$ and this operator has a simple and isolated eigenvalue which is $\gamma(t_0, \lambda_0) = e^{P_{\lambda^0}(t_0)}$, where $P_{\lambda^0}(t_0)$ is the topological pressure of f_{λ^0} at t_0 (see Proposition 9.6). From the perturbation theory for linear operators (see [Ka]) it follows now that there is $r > 0$ and a holomorphic map

$$(t, \lambda) \in \mathbb{D}_{\mathbb{C}}(t_0, r) \times \mathbb{D}_{\mathbb{C}^{2d}}(\lambda^0, r) \mapsto \gamma(t, \lambda)$$

such that

1. $\gamma(t, \lambda)$ is a simple isolated eigenvalue of $L(t, \lambda) \in L(H_\tau)$ and

(2) there is $\beta > 0$ such that the spectrum

$$\sigma(L(t,\lambda)) \cap \mathbb{D}(e^{P_{\lambda^0}(t_0)}, \beta) = \{\gamma(t,\lambda)\}$$

for all $(t,\lambda) \in \mathbb{D}_{\mathbb{C}}(t_0, r) \times \mathbb{D}_{\mathbb{C}^{2d}}(\lambda^0, r)$.

Coming now back to the initial parameters, real t and $\lambda \in \mathbb{C}^d$, we remember that the operators $L(t,\lambda)$ are conjugate to $\mathcal{L}_{t,\lambda}$ via the operator T_λ that consist in composition with the holomorphic motion h_λ. From the Hölder continuity property (Proposition 9.3) of h_λ it follows that we may assume that there is $0 < \tau \leq 1$ such that $T_\lambda(H_1(\mathcal{J}(f_\lambda), \mathbb{C})) \subset H_\tau(\mathcal{J}(f_{\lambda^0}), \mathbb{C})$ for all $\|\lambda - \lambda_0\| < r$. Consequently $e^{P_\lambda(t)}$, $P_\lambda(t)$ the topological pressure of f_λ at t, is an eigenvalue of $\mathcal{L}_{t,\lambda}$ provided we can show the following:

Lemma 9.10. *For every $t > \rho/\alpha$ the function $\lambda \mapsto \mathrm{P}_\lambda(t)$ is continuous*

Proof. We have that $\mathrm{P}_\lambda(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{z\in f_\lambda^{-1}(w)} |(f_\lambda^n)'(z)|_\sigma^{-t}$ with $w \in \mathcal{J}(f_\lambda)$ is any finite point. The continuity assertion results directly from Lemma 9.7 since it is shown there that for any $z \in \mathcal{J}_0$

$$(1-\varepsilon)^n \leq \frac{|(f_\lambda^n)'(h_\lambda(z))|_\sigma}{|(f_{\lambda^0}^n)'(z)|_\sigma} \leq (1+\varepsilon)^n.$$

□

Altogether we obtained real analyticity of the pressure function. From Bowen's formula (Theorem 1.3)) we know that the hyperbolic dimension $\mathrm{HD}(\mathcal{J}_r(f_\lambda))$ is the only zero of the pressure function $t \mapsto \mathrm{P}_\lambda(t)$. Real analyticity of this zero with respect to λ results from the implicit function theorem since clearly

$$\frac{\partial}{\partial t}\mathrm{P}_\lambda(t) \leq -\log\rho < 0$$

where $\rho > 1$ is the expanding constant that is common to the f_λ (see Fact 2.3).

10. Around Theorem 1.6

In this section we derive the most transparent consequences of Theorem 1.7, notably Theorem 1.6. We begin with the following.

Theorem 10.1. *Let $f_\lambda = f \circ P_\lambda$ with $f : \mathbb{C} \to \hat{\mathbb{C}}$ a meromorphic function and, for every $\lambda = (\lambda_d, \lambda_{d-1}, \dots, \lambda_1, \lambda_0) \in \mathbb{C}^{d+1}$, $P_\lambda : \mathbb{C} \to \mathbb{C}$ is the polynomial given by the formula $P_\lambda(z) = \sum_{j=0}^d \lambda_j z^j$. Suppose that f_{λ^0} is hyperbolic and that there is a neighborhood $U \subset \mathbb{C} \setminus \{0\} \times \mathbb{C}^d$ of λ^0 such that $\{f_\lambda;\ \lambda \in U\}$ is uniformly balanced with $\alpha_2 \geq 1$ and $\alpha_1 \geq 0$. Then the function $\lambda \mapsto \mathrm{HD}(\mathcal{J}_r(f \circ P_\lambda))$ is real-analytic near λ^0.*

Proof. Put $f_\lambda = f\circ P_\lambda$. For every $\gamma\in\mathbb{C}^{d+1}$ put

$$Q_\gamma(z) = z^d + \sum_{j=0}^{d-1}\gamma_j\gamma_d^{-j}z^j \quad\text{and}\quad g_\gamma = \gamma_d f\circ Q_\gamma.$$

Consider also H, the change of coordinates in the parameter space, given by the formula

$$H(\lambda_d,\lambda_{d-1},\dots,\lambda_1,\lambda_0) = (\lambda_d^{1/d},\lambda_{d-1},\dots,\lambda_1,\lambda_0),$$

where $\lambda_d\mapsto\lambda_d^{1/d}$ is a holomorphic branch of dth radical defined on the ball $\mathbb{D}_{\mathbb{C}^d}(\lambda_d^0,|\lambda_d^0|)$. Let $T_\gamma:\mathbb{C}\to\mathbb{C}$ be the multiplication map defined as $T_\gamma(z)=\gamma_d^{-1}z$. Notice that

$$T_{H(\lambda)}\circ g_{H(\lambda)}\circ T^{-1}_{H(\lambda)} = f_\lambda. \tag{10.1}$$

So, $\mathcal{J}_r(f_\lambda) = T_{H(\lambda)}(\mathcal{J}_r(g_{(H(\lambda)}))$, and in consequence,

$$\mathrm{HD}(\mathcal{J}_r(f_\lambda)) = \mathrm{HD}(\mathcal{J}_r(g_{(H(\lambda)})).$$

Since in addition $H(\lambda^0) = \big((\lambda_d^0)^{1/d},\lambda_{d-1}^0,\dots,\lambda_1^0,\lambda_0^0\big)$, in order to prove our theorem, it is enough to show that the map $\gamma\mapsto\mathrm{HD}(\mathcal{J}_r(g_\gamma))$ is real-analytic near the point $\gamma^0 = \big((\lambda_d^0)^{1/d},\lambda_{d-1}^0,\dots,\lambda_1^0,\lambda_0^0\big)\in H(U)$. It follows from (10.1) that for every λ in a neighbourhood of λ^0 and every $z\in\mathcal{J}(g_{(H(\lambda)}))$, we have

$$|g_{H(\lambda)}(z)| = |\lambda_d^{1/d}||f_\lambda(T_{H(\lambda)}(z))| \quad\text{and}\quad |g'_{H(\lambda)}(z)| = |f'_\lambda(T_{H(\lambda)}(z))|.$$

Consequently,

$$\frac{|g'_{H(\lambda)}(z)|}{|g_{H(\lambda)}(z)|} = |\lambda_d^{-\frac{1}{d}}|\frac{|f'_\lambda(T_{H(\lambda)}(z))|}{|f_\lambda(T_{H(\lambda)}(z))|}.$$

Since $T_{H(\lambda)}(z)\in\mathcal{J}(f_\lambda)$ and since $|\lambda_d^{-\frac{1}{d}}|$ is bounded away from zero and infinity on a neighbourhood of λ^0, it follows from the uniform balanced growth of $\{f_\lambda;\,\lambda\in U\}$ that g_λ also has this property for λ near λ^0. Aiming to apply Theorem 1.7, we are therefore left to show that for a sufficiently small bounded neighbourhood of γ^0, the family $\mathcal{M}_U = \{g_\gamma\}_{\gamma\in H(U)}$ is of bounded deformation. We have for every $z\in\mathbb{C}$ that

$$g'_\gamma(z) = \gamma_d f'(Q_\gamma(z))Q'_\gamma(z) = \gamma_d f'(Q_\gamma(z))\left(dz^{d-1} + \sum_{j=1}^{d-1} j\gamma_j\gamma_d^{-1}z^{j-1}\right), \tag{10.2}$$

$$\begin{aligned}\dot{g}_d(\gamma,z) &:= \frac{\partial g_\gamma}{\partial\gamma_d}(z) = f(Q_\gamma(z)) + \gamma_d f'(Q_\gamma(z))\frac{\partial Q_\gamma(z)}{\partial\gamma_d},\\ &= f(Q_\gamma(z)) + \gamma_d f'(Q_\gamma(z))\sum_{j=1}^{d-1} -j\gamma_j\gamma_d^{-j-1}z^j,\end{aligned} \tag{10.3}$$

and

$$\dot{g}_i(\gamma,z) := \frac{\partial g_\gamma}{\partial\gamma_i}(z) = \gamma_d f'(Q_\gamma(z))\frac{\partial Q_\gamma(z)}{\partial\gamma_i} = \gamma_d f'(Q_\gamma(z))\gamma_d^{-i}z^i \tag{10.4}$$

for all $i = 0, 1, \dots, d-1$. Taking U sufficiently small, there clearly exists $p \in (0, +\infty)$ such that

$$|Q'_\gamma(z)| \geq 1 \tag{10.5}$$

for all $\gamma \in H(U)$ and all $z \in \mathbb{C}$ with $|z| \geq p$. Now, since g_γ is of uniformly rapid derivative growth on $H(U)$ and since, after a conjugation by translation, there exists $R > 0$ such that

$$\mathcal{J}(g_\gamma) \cap D(0, R) = \emptyset \tag{10.6}$$

for all $\gamma \in H(U)$, it follows from (10.2) that $Q'_\gamma(z) \neq 0$ for all $\gamma \in H(U)$ and all $z \in \mathcal{J}(g_\gamma)$. By a standard compactness argument, it then follows from J-stability of g_{γ_0} that decreasing U appropriately, we get

$$A := \inf\{|Q'_\gamma(z)| : \gamma \in H(U),\ z \in \overline{D}(0, p) \cap \mathcal{J}(g_\gamma)\} > 0.$$

Combining this and (10.5), we obtain

$$B := \inf\{|Q'_\gamma(z)| : \gamma \in H(U),\ z \in \mathcal{J}(g_\gamma)\} \geq \min\{1, A\} > 0.$$

It follows from (10.2) and (10.4) that for all $i = 0, 1, \dots, d-1$ we have

$$\frac{|\dot{g}_i(\gamma, z)|}{|g'_\gamma(z)|} = |\gamma_d|^{-i} \frac{|z|^i}{|Q'_\gamma(z)|} = |\gamma_d|^{-i} \frac{|z|^i}{\left|dz^{d-1} + \sum_{j=1}^{d-1} j\gamma_j \gamma_d^{-1} z^{j-1}\right|}. \tag{10.7}$$

Since obviously, $\lim_{z\to\infty}(|z|^i/|Q'_\gamma(z)|) \leq 1/d$ uniformly with respect to $\gamma \in H(U)$ for all $i = 0, 1, \dots, d-1$, invoking the definition of B, we see that

$$B_1 := \max_{0 \leq i \leq d-1} \left\{ \sup \left\{ \frac{|z|^i}{|Q'_\gamma(z)|} : \gamma \in H(U),\ z \in \mathcal{J}(g_\gamma) \right\} \right\} < +\infty.$$

Combining this and (10.7), we see that with U sufficiently small,

$$B_2 := \max_{0 \leq i \leq d-1} \left\{ \sup \left\{ \frac{|\dot{g}_i(\gamma, z)|}{|g'_\gamma(z)|} : \gamma \in H(U),\ z \in \mathcal{J}(g_\gamma) \right\} \right\} < +\infty. \tag{10.8}$$

It follows from (10.2) and (10.3) that

$$\begin{aligned} \frac{|\dot{g}_d(\gamma, z)|}{|g'_\gamma(z)|} &:= \left| \frac{f \circ Q_\gamma(z)}{\gamma_d (f \circ Q_\gamma)'(z)} + \frac{\frac{\partial Q_\gamma(z)}{\partial \gamma_d}}{|Q'_\gamma(z)|} \right| \\ &\leq |\gamma_d|^{-1} \frac{|f \circ Q_\gamma(z)|}{|(f \circ Q_\gamma)'(z)|} + \left| \frac{\sum_{j=1}^{d-1} -j\gamma_j \gamma_d^{-j-1} z^j}{dz^{d-1} + \sum_{j=1}^{d-1} j\gamma_j \gamma_d^{-1} z^{j-1}} \right|. \end{aligned} \tag{10.9}$$

Since we have the uniformly balanced growth property, since $\alpha_1 \geq 0$ and $\alpha_2 \geq 1$, and taking into account (10.6), we conclude that

$$B_3 := \sup \left\{ |\gamma_d|^{-1} \frac{|f \circ Q_\gamma(z)|}{|(f \circ Q_\gamma)'(z)|} : \gamma \in H(U),\ z \in \mathcal{J}(g_\gamma) \right\} < +\infty. \tag{10.10}$$

Since obviously,

$$\lim_{z\to\infty} \sup_{\gamma\in H(U)} \left\{ \left| \frac{\sum_{j=1}^{d-1} -j\gamma_j\gamma_d^{-j-1}z^j}{dz^{d-1}+\sum_{j=1}^{d-1} j\gamma_j\gamma_d^{-1}z^{j-1}} \right| \right\} < +\infty,$$

invoking the definition of B, we see that

$$B_4 := \sup\left\{ \left| \frac{\sum_{j=1}^{d-1} -j\gamma_j\gamma_d^{-j-1}z^j}{dz^{d-1}+\sum_{j=1}^{d-1} j\gamma_j\gamma_d^{-1}z^{j-1}} \right| : \gamma\in H(U),\ z\in\mathcal{J}(g_\gamma) \right\} < +\infty.$$

Combining this, (10.10), (10.9), and (10.8), we see that

$$\max_{0\le i\le d}\left\{ \sup\left\{ \frac{|\dot g_i(\gamma,z)|}{|g_\gamma'(z)|} : \gamma\in H(U),\ z\in\mathcal{J}(g_\gamma) \right\} \right\} < +\infty.$$

We are done. $\square$

Note that if $d = 1$, then with the notation of the proof of the previous theorem, $g_{\lambda_d} = \lambda_d f$ and, as an immediate consequence of this proof, we have the following.

Corollary 10.2. *Suppose that $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a meromorphic function and consider the analytic family $\mathcal{F} = \{\lambda f\}_{\lambda\in\mathbb{C}\setminus\{0\}}$. If $f \in \mathcal{F}$ is hyperbolic and if this family is uniformly balanced near f with $\alpha_2 \ge 1$ and $\alpha_1 \ge 0$, then the function $\lambda \mapsto \mathrm{HD}(\mathcal{J}_r(\lambda f))$ is real-analytic in a neighbourhood of $\lambda^0 = 1$.*

Remark 10.3. *If in the formulation of Theorem 10.1 the parameter λ_d is kept fixed equal to 1, then the derivative $\dot g_d(\gamma, z)$ disappears and it suffices to assume that $\alpha_2 > 0$ (and $\alpha_1 \ge 0$).*

We end this section by noting that Theorem 1.6 is an immediate consequence of Theorem 10.1 and Lemma 3.2.

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